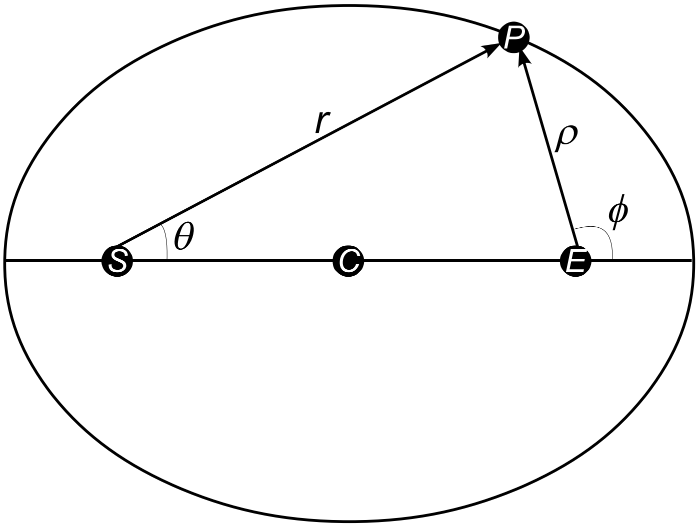
Here again is the figure introducing the variables (r, θ) and (ρ, φ).



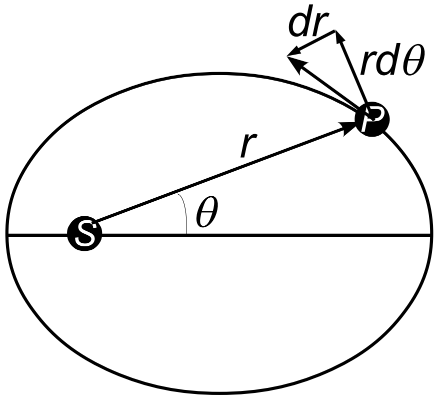
Also v is the linear speed.

The *apsides* are the locations of minimum and maximum distance, r_{min} and r_{max}. Let v_{min} be the linear speed at r_{min} and likewise for v_{max}.

We will compare angular speeds, that is, $\dot{\theta}$.

K is the constant $\dot{\varphi}$ in the equant law. K_A is the constant $\dot{A}$ in the area law. K_I is the constant in the formula $v = K_I/r$ in the inverse distance law.

In the general case (i.e., not just at the apsides) note that the linear velocity of P has tangential component $r\dot{\theta}$ and radial component $\dot{r}$, thus linear speed $\sqrt{r^2\dot{\theta}^2 + \dot{r}^2}$ in the (r, θ) system. See this figure:



In the (ρ, φ) system, the linear speed is of course $\sqrt{\rho^2\dot{\varphi}^2 + \dot{\rho}^2}$. Since the linear speed is the same in both cases, we have:

$$\begin{aligned}
\sqrt{r^2\dot{\theta}^2 + \dot{r}^2} &= \sqrt{\rho^2\dot{\varphi}^2 + \dot{\rho}^2} \\
r^2\dot{\theta}^2 + \dot{r}^2 &= \rho^2\dot{\varphi}^2 + \dot{\rho}^2 \\
\dot{\theta}^2 &= \frac{\rho^2\dot{\varphi}^2 + \dot{\rho}^2 - \dot{r}^2}{r^2} \\
\dot{\theta} &= \frac{\sqrt{\rho^2\dot{\varphi}^2 + \dot{\rho}^2 - \dot{r}^2}}{r}
\end{aligned}$$

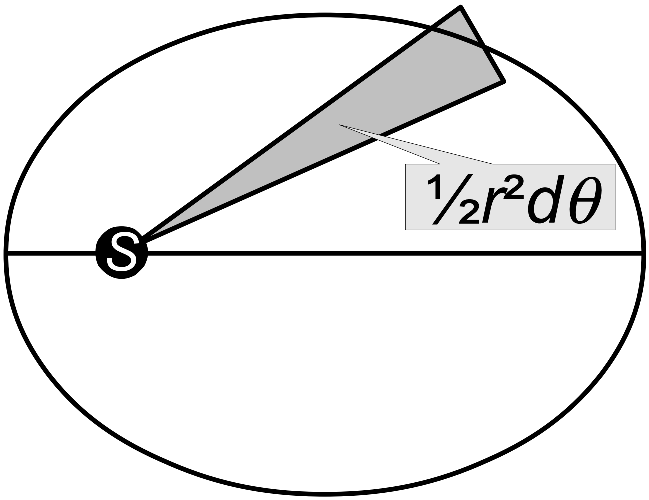
In the special case of an elliptical orbit, $r + \rho =$ constant, so $\dot{r} = -\dot{\rho}$ and $\dot{r}^2 = \dot{\rho}^2$. We get a much simpler formula, using also $\dot{\varphi} = K$:

$$\dot{\theta} = \frac{K\rho}{r} \quad \text{Equant law for ellipse}$$

We could even eliminate ρ, using $r + \rho = 2a$, but that won't be necessary.

For Ptolemy's eccentric circular orbits, we would have to use the messier form. But for small eccentricities, the difference will be small.

For the area law, let A be the area swept out by the radius vector from some fixed starting point. The infinitesimal increment of area dA in time dt is $\frac{1}{2}r^2 d\theta$, as in the this figure:



Thus the constant rate of increase is $\dot{A} = \frac{1}{2}r^2\dot{\theta}$. So

$$\frac{1}{2}r^2\dot{\theta} = K_A$$

$$\dot{\theta} = \frac{2K_A}{r^2} \quad \text{Area law}$$

Finally, we have Kepler's inverse distance law, where $v = K_I/r$. We have

$$\begin{aligned}
\sqrt{r^2\dot{\theta}^2 + \dot{r}^2} &= \frac{K_I}{r} \\
r^2\dot{\theta}^2 + \dot{r}^2 &= \frac{K_I^2}{r^2} \\
\dot{\theta}^2 &= \frac{K_I^2}{r^4} - \frac{\dot{r}^2}{r^2} \\
&= \frac{K_I^2 - r^2\dot{r}^2}{r^4} \\
\dot{\theta} &= \frac{\sqrt{K_I^2 - r^2\dot{r}^2}}{r^2} \quad \text{Inverse distance law}
\end{aligned}$$

This is a bit messy. It takes on a simpler form at the apsides, where $\dot{r} = 0$:

$$\dot{\theta} = \frac{K_I}{r^2} \quad \text{Inverse distance law at apsides}$$

For an orbit with small eccentricity (whether circular or elliptical), $\dot{r}$ is small throughout, so this simpler form is approximately true everywhere.

Putting all three forms together:

$$\begin{aligned}
&\dot{\theta} = \tfrac{Kr\rho}{r^2} && \text{Equant law for ellipse} \\
&\dot{\theta} = \tfrac{2K_A}{r^2} && \text{Area law} \\
&\dot{\theta} = \tfrac{K_I}{r^2} && \text{Inverse distance law at apsides}
\end{aligned}$$

As noted in the post, I've made the denominator r^2 in all three laws to facilitate comparison.