Bundles

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This is a brief note on fiber bundles and principal bundles. The sole purpose is to clarify the relation between a principal bundle and its associated bundles. No formal proofs.

First we work through the case of the Möbius band, then the tangent bundle of the 2-sphere. We describe how to go from the fiber bundle to the principal bundle, and back again.

Fiber Bundle to Principal Bundle

Let (X, B, π, F, G) be the band, where:

1. X is the band (a topological space).
2. B is the base space, which is the circle S^1; we can think of B as the centerline of the band.
3. $\pi : X \to B$ is the projection map.
4. F is the fiber, which is the interval $[-1, 1]$. So for each $b \in B$, $\pi^{-1}(b)$ is homeomorphic to F, but *not canonically.*
5. G is the structure group, namely $\mathbb{Z}_2 = \{1, -1\}$, which acts on F in the obvious way: 1 is the identity mapping, and -1 sends s to $-s$. We write gs for this action ($g \in G, s \in F$). We make G into a topological group by giving it the discrete topology.

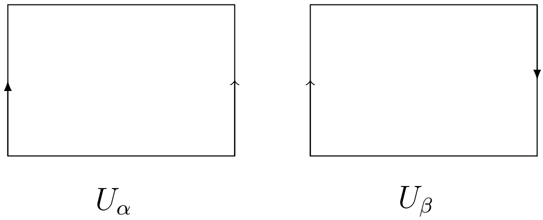
Next: the principal bundle (P, B, τ, G), where:

1. P is the total space of bundle.
2. B is the base space.
3. τ is the projection map.
4. G is both the group and the fiber.

The fibers $\tau^{-1}(b)$ are all homeomorphic to G (but *not canonically*), so P is a double cover of B. The idea is to make P look as much like X as possible, given the different fiber. There are exactly two double covers of B: the trivial one and the one given by $z \mapsto z^2$ on the unit circle. The latter is the one we want; we can think of it as the edge of the band, which wraps twice around the centerline.

To construct P, we imagine that the band X is constructed by taking two rectangular strips of

Figure 1: Möbius Band



Möbius Band obtained by gluing two rectangular strips together. The inner two vertical edges are glued together without a twist, the outer two vertical edges with a twist. In the formal treatment, not just vertical edges but vertical open "zones" are glued (identified), since U_α and U_β are open sets, and the α and β representations of $\{b\} \times [-1, 1]$ are identified for any $b \in U_\alpha \cap U_\beta$.

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paper and gluing them together in the right way (i.e., with a twist: see fig.1). P is constructed almost as a byproduct, since each strip contains a top edge and a bottom edge, and these four edges get glued into a single circle as the two strips are glued together to make the band.

More formally, we cover B with open sets $\{U_\alpha : \alpha = 1, 2\}$ which "trivalize" the bundle X, i.e., for which $\pi^{-1}(U_\alpha)$ is homeomorphic to $U_\alpha \times F$. (So those are the two strips.) Also, these homeomorphisms preserve fibers: if $\varphi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times F$ is one of the homeomorphisms, then φ_α maps $\pi^{-1}(b)$ to the fiber over b in $U_\alpha \times F$, i.e., $\{b\} \times F$.

So if $b \in U_\alpha \cap U_\beta$ $(\alpha \neq \beta)$, then φ_α and φ_β both make $\pi^{-1}(b)$ homeomorphic to F. But in two different ways: $\varphi_\beta \varphi_\alpha^{-1}$ induces a homeomorphism of F to itself, and this composition is either the identity or the map $s \mapsto -s$. In other words, it's given by one of the actions of

G on F. (We say “induces” because technically, $\varphi_\beta\varphi_\alpha^{-1}$ maps $\{b\} \times F$ to itself, a fine point we will ignore from now on.)

This formal description assumes we already have X in hand, otherwise we can't speak of $\pi^{-1}(U_\alpha)$. Let's make the math follow the arts and crafts more closely. Start with the spaces $\{U_\alpha \times F : \alpha = 1, 2\}$, i.e., the two strips of paper shown in fig.1. Take the disjoint union. For each $b \in U_\alpha \cap U_\beta$, we chose an element of G, call it $g_{\alpha\beta}(b)$. Moreover, let's make $g_{\alpha\beta}$ continuous. Then $g_{\alpha\beta}$ tells us how to glue the strips together. Formally, we identify (b, s) as an element of $U_\alpha \times F$ with $(b, g_{\alpha\beta}(b)s)$ as an element of $U_\beta \times F$. (Recall that gs stands for the action of G on F.) The $g_{\alpha\beta}$ work as gluing instructions, though they are called transition functions.

Now for the punchline: we can do the same thing if we replace the fiber F with G as a fiber. We start off with the disjoint union of spaces

$\{U_\alpha \times G : \alpha = 1, 2\}$ — that's the four edges. The transition functions $g_{\alpha\beta}$ restrict to transition functions for G, so we know how to glue the four edges together.

Time to generalize. Let (X, B, π, F, G) be any old fiber bundle. So we have an open cover $\{U_\alpha\}$ of B, with trivializing homeomorphisms $\varphi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times F$ preserving fibers. Moreover each transition function $\varphi_\beta \varphi_\alpha^{-1} : F \to F$ is given by the action of G on F, according to the formula

$$\varphi_\beta \varphi_\alpha^{-1}(s) = g_{\alpha\beta}(b)s, \quad b = \pi(s)$$

where $g_{\alpha\beta} : U_\alpha \cap U_\beta \to G$ is a continuous function. So we can form the disjoint union of the $\{U_\alpha \times F\}$, and then use the $g_{\alpha\beta}$ to glue together (i.e., identify) various fibers.

Next, the principal bundle: we form the disjoint union of the $\{U_\alpha \times G\}$. Potential problem: there's no reason to believe that G will be a subset of F. So we can't "restrict" the $g_{\alpha\beta}$ to G. But this problem dissolves when you realize that $g_{\alpha\beta}(b)$ is an element of G, so it acts on G by multiplication (on the left).

Let's do another example: the tangent bundle of S^2. The fibers $T_b(S^2)$ are all homeomorphic to $\mathbb{R}^2$; indeed they are linearly isomorphic. So set $F = \mathbb{R}^2$. For each U_α we flatten the open set diffeomorphically onto the plane; this gives linear isomorphisms $\varphi_\alpha(b) : T_b(S^2) \to \mathbb{R}^2$ for each $b \in U_\alpha$ (the derivative maps). (As before, we will be cavalier about the distinction between $T_b(S^2) \to \mathbb{R}^2$ and $T_b(S^2) \to \{b\} \times \mathbb{R}^2$.)

It's pretty clear that the transition functions $\varphi_\beta\varphi_\alpha^{-1}$ are elements of $GL_{\mathbb{R}}(2)$, thus $g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL_{\mathbb{R}}(2)$. In fact, the $g_{\alpha\beta}$ are all smooth, but let's not get into that.

Now we form the principal bundle. This is often called the frame bundle. Here's the idea: if

$[\hat{x}, \hat{y}]$ is the canonical basis of $\mathbb{R}^2$, then we have a canonical 1–1 correspondence between $GL_{\mathbb{R}}(2)$ and the set of bases of $\mathbb{R}^2$, namely $g \leftrightarrow [g\hat{x}, g\hat{y}]$. (We use brackets to indicate an ordered basis.) So the elements of the fiber can be viewed as "frames".

Let's try to picture an element t of the principal bundle above $b \in S^2$. From our definitions, it's an equivalence class: for each $U_\alpha \ni b$, we have a "representative" of t in $U_\alpha \times GL_{\mathbb{R}}(2)$, call it $[\hat{x}_\alpha, \hat{y}_\alpha]$. Representatives are related by the transition functions, and t is the equivalence class of all the representatives.

However, we can give a more concrete description. We have a linear isomorphism $\varphi_\alpha(b)$: $T_b(S^2) \to \mathbb{R}^2$. That sets up a 1–1 correspondence between bases in $T_b(S^2)$ and bases in $\mathbb{R}^2$. The same goes, *mutatis mutandis*, for φ_β. Now

$g_{\alpha\beta}(b)$ is given by a composition:

$$[\hat{x}_\alpha, \hat{y}_\alpha] \xrightarrow{\varphi_\alpha^{-1}(b)} [\hat{x}_t, \hat{y}_t] \xrightarrow{\varphi_\beta(b)} [\hat{x}_\beta, \hat{y}_\beta]$$

where $[\hat{x}_t, \hat{y}_t] \subset T_b(S^2)$ is a basis "living on the sphere". Each equivalence class is represented in a natural way by a basis of $T_b(S^2)$.

Obviously we have been way too specific, and this discussion works for any differential n-manifold. If the manifold has more structure, we can replace $GL(n)$ with other groups. For example, if it's a Riemannian manifold, then we can restrict ourselves to orthonormal bases, and replace $GL(n)$ with $O(n)$, the orthogonal group.

Principal Bundle to Fiber Bundle

Suppose we have a principal bundle (P, B, τ, G). How can we replace the fiber G with another fiber F? Answer: if we have a left action of G on F, say $(g, s) \mapsto gs$, then we can define the so-called associated bundle (X, B, π, F, G).

The definition is rather simple: just let $\tilde{X} = P \times F$, and identify:

$$(p, s) \equiv (pg^{-1}, gs) \quad \forall g \in G$$

and let X be $\tilde{X}/\equiv$. I omit the rest of the construction.

The fiber will be F. I like to look at it this way: we start with a point $b \in B$. For each $p \in P$ in the τ-fiber over b, we have a *coordinate system* for the π-fiber over b. Coordinates are elements of F; the coordinate of $(p, s)/\equiv$ is just s, using the coordinate system associated with p.

Let's look at the Möbius band. The principal bundle looks like the edge, a double covering of the centerline. The τ-fiber over b is a pair of points. Now we want to construct the band. Here the fibers will be homeomorphic to $F = [-1, 1]$. Given a $b \in B$, we have two coordinate systems for $\pi^{-1}(b)$, as we can lay out $[-1, 1]$ along $\pi^{-1}(b)$ going either way. A coordinate s

using one system becomes $-s$ using the other system. Thus $-1 \in G$ will take us from one edge to the other, and will adjust the coordinate accordingly.

The tangent space example is also not too difficult. The elements of the τ-fiber can be viewed as bases of $T_b(S^2)$, as noted. If t is such a basis, then we have a coordinate system for $T_b(S^2)$ associated with it. Coordinates are elements of the fiber $\mathbb{R}^2$, i.e., pairs of real numbers. A change of basis results in a change of coordinates, and "contravariantly". For example, if we go from $[\hat{x}, \hat{y}]$ to $[2\hat{x}, \hat{y}]$, then the coordinates (x, y) go to $(\frac{1}{2}x, y)$. That's why we have (pg^{-1}, gs) in the definition of $\equiv$.

Torsors and the Principal Bundle

When constructing the principal bundle, we started with the disjoint union of the $U_\alpha \times G$, and then identified elements. In the Möbius band, these

$U_\alpha \times G$ were two pairs of edges, which glued together became the edge of the band.

There's something a bit odd about this: the group $G = \{1, -1\}$ has a distinguished element, namely the identity. However, the two edges of a pair "look the same". To reflect this, we should really start with the disjoint union of $U_\alpha \times T$, where T has the same cardinality as G, but the bijection between T and G is *not canonical.*

The notion needed is a *torsor.* T is a torsor for G if:

1. There is a right action of G on T: $(t, g) \mapsto tg$.
2. The action is transitive: for any $s, t \in T$, there is a g such that $sg = t$.
3. The action is free: except for $1 \in G$, the

action is without fixed points. So $(\exists t) tg = t \Rightarrow g = 1$.

It follows that as soon as we pick an element $t_0 \in T$, we have a 1–1 correspondence between G and T, $g \mapsto t_0 g$.

Accordingly, we write (P, B, τ, G, T) for a principal bundle, replacing G with T as the "generic fiber".

The prototypical example of a torsor is an oriented circle: the rotation group $U(1)$ acts on it. In contrast, the sphere S^2 is not a torsor for the rotation group $SO(3)$, since the action is not free.

Any set with two elements is a torsor for $\mathbb{Z}_2$, with a uniquely determined action. This is the torsor we need for the Möbius band.

Let V be an finite-dimensional real vector space.

The set of bases for V (call it T) is a torsor for $GL(V)$. It is slightly less obvious, but T is also a torsor for $GL_{\mathbb{R}}(n)$, if $\dim V = n$. How do we define the action of $g \in GL_{\mathbb{R}}(n)$ on a basis $[v_1, \ldots, v_n]$? Answer: $[v_1, \ldots, v_n]$ determines a linear isomorphism between V and $\mathbb{R}^n$, call it $q : V \to \mathbb{R}^n$. Then $q^{-1}gq$ is in $GL(V)$, so we can apply it to $[v_1, \ldots, v_n]$. In matrix terms: g has a canonical representation in $\mathrm{Mat}_{\mathbb{R}}(n \times n)$ (just use the standard basis), and we can apply the same matrix to $[v_1, \ldots, v_n]$.

Example: if $g \in GL_{\mathbb{R}}(2)$ is defined by $[\hat{x}, \hat{y}]g = [2\hat{x}, \hat{y}]$, then for any basis $[v_1, v_2]$ of V, we let $[v_1, v_2]g = [2v_1, v_2]$.

This highlights a key property of principal bundles, not shared with more general bundles: we have an action of G on the entire space P. Not so for (X, B, π, G, F). We have an action of G on the generic fiber F, but no way of applying this "across the board", since the homeomorphisms between $\pi^{-1}(b)$ and F are not canonical.

For example, compare the tangent with the frame bundle on S^2. $G = GL_{\mathbb{R}}(2)$. As noted, we know how to apply $g \in G$ to any basis of $T_p(S^2)$. But if we are looking at a vector $v \in T_p(S^2)$, we don't generally know how to apply g to v, as illustrated by $[v_1, v_2] \mapsto [2v_1, v_2]$.