Hatcher Notes

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These notes discuss items in Hatcher's *Algebraic Topology* I puzzled over at some point, or sometimes just add observations I find helpful. All page references are to Hatcher's text (the online version with the 2002 copyright on the title page).

1 §2.1: Relative Homology

p.115: Start with the long exact sequence:

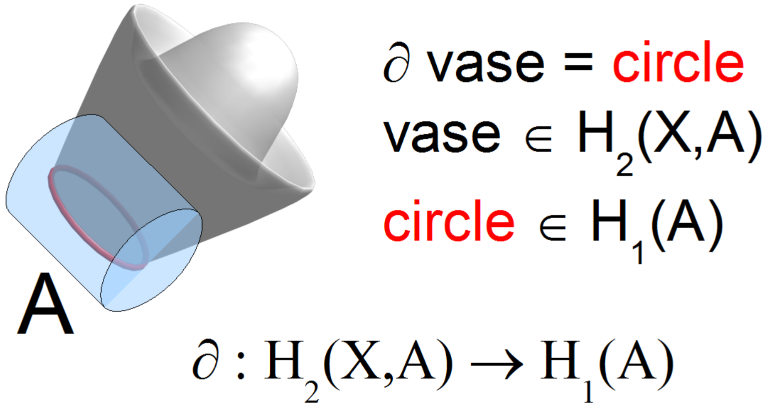
$$\cdots \to H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X,A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

or the reduced long exact sequence

$$\cdots \to \tilde{H}_n(A) \xrightarrow{i_*} \tilde{H}_n(X) \xrightarrow{j_*} \tilde{H}_n(X,A) \xrightarrow{\partial} \tilde{H}_{n-1}(A) \to \cdots \tag{1}$$

Provided $A \neq \varnothing$, these sequences are identical except for $H_0(A)$, $H_0(X)$, and the corresponding reduced groups. These four groups are all free abelian; the rank of $H_0(A)$ equals the number of path components of A, while the rank

Figure 1: The Connecting Homomorphism



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of $\tilde{H}_0(A)$ is one less, and likewise for X (see p.109–110). $H_n(A) = \tilde{H}_n(A)$ for $n > 0$, likewise for X, and $H_0(X, A) = \tilde{H}_0(X, A)$.

Let's look first at the special case where X is path-connected and $H_1(X) = 0$. Then from (1), we get:

$$0 \to H_1(X, A) \to \tilde{H}_0(A) \to 0$$

that is, $H_1(X, A)$ is isomorphic to $\tilde{H}_0(A)$, hence free abelian with rank one less than the number of path components of A. As a concrete example, consider $X = \mathbb{R}$ and $A = \{1, 2\}$. So $H_1(\mathbb{R}, \{1, 2\}) = \mathbb{Z}$. For A a finite set with k elements, $H_1(\mathbb{R}, A) = \mathbb{Z}^{(k-1)}$.

Fig. 1 illustrates the connecting homomorphism $\partial : H_n(X, A) \to H_{n-1}(A)$ for the case $n = 2$. The vase stands for a chain representing an element of $H_2(X, A)$; A is the solid cylinder. The boundary of the vase (the red circle) lies in A, hence the vase is a relative cycle. Its boundary, the circle, determines an element of $H_1(A)$.

Next we drill down into the details of the connecting homomorphism . We match our notation to Hatcher's diagram on p.116. Hatcher treats the general case of a short exact sequence $0 \to A_n \to B_n \to C_n \to 0$; specialized to $0 \to C_n(A) \to C_n(X) \to C_n(X,A) \to 0$, we have:

$$\begin{tikzcd}
 & a \in C_{n-1}(A) \arrow[d, "i"] \\
b \in C_n(X) \arrow[r, "\partial"] \arrow[d, "j"] & \partial b \in C_{n-1}(X) \\
c \in C_n(X,A) &
\end{tikzcd} \tag{2}$$

We have two sorts of equivalence classes to contend with: homology classes and $C_n(A)$-cosets. Starting at the bottom, $c \in C_n(X,A)$ is a representative element of a homology class $[c] \in H_n(X,A)$. Now, c itself is a $C_n(A)$-coset, with a representative element $b \in C_n(X)$. Then ∂b is a chain in $C_{n-1}(X)$. Not just any chain: c must be a cycle (by definition of H_n), so $\partial c = 0$, so ∂b must belong to the 0-coset, i.e., ∂b must be in $C_{n-1}(A)$. We let $a = \partial b$, and define the connecting homomorphism by $\partial[c] = [a]$, where $[a]$ is the homology class of a. Hatcher shows this is independent of the various choices made.

What does this mean for our example $H_1(\mathbb{R}, \{1,2\})$? Here b is a 1-chain in $\mathbb{R}$, roughly a collection of paths, more precisely an integral linear combination of 1-simplices. We can add or delete simplices lying entirely in $\{1,2\}$ without changing the coset $c = b + C_1(\{1,2\})$. In order for c to be a cycle, ∂b must lie in $\{1,2\}$. In other words, $\partial b = m1 + n2$ for some integers m and n, where 1 and 2 denote the obvious 0-simplices. We must have $m = -n$: for in general, if $\partial p = \sum_i m_i q_i$ for a 1-chain p, then $\sum_i m_i = 0$. (See Hatcher p.109–110, the discussion of reduced homology.) So if e is a 1-simplex going from 1 to 2 (i.e., $\partial e = 2 - 1$), then e determines the basis element for the group $H_1(\mathbb{R}, \{1,2\}) = \mathbb{Z}$. ("Determines" means "first pass to the $C_1(A)$-coset, then take the homology class of that".)

A bit more generally, if A has k elements, $A = \{a_1, \ldots, a_k\}$, then we obtain a basis for $H_1(\mathbb{R}, A)$ with $k-1$ simplices e_i, $i = 2, \ldots k$, with e_i going from a_1 to a_i. For if b has $\partial b \in C_0(A)$, we can form a linear combination of the e_i with the same boundary. The difference will be a true cycle, hence homologous to 0 in $\mathbb{R}$.

Let's see what happens to this basis when we hit it with the connecting homomorphism. Since $\partial e_i = a_i - a_1$ (using a_i and a_1 to denote the obvious 0-chains), we see that we get a basis for the reduced homology group $\tilde{H}_0(A)$.

Two more examples: the cylinder, and the Möbius strip. $H_n(\text{cyl}) = H_n(\text{Möb})$ for all n, since both are homotopic to the circle S^1. Thus $H_1 = H_0 = \mathbb{Z}$, and all other $H_n = 0$. Interesting differences emerge when we look at the relative groups. I will not attempt full rigor, just intuition. Also, instead of dealing with homology and equivalence classes, I will deal directly with chains in the cylinder and Möbius strip.

First, a historical digression. A key question in the early years of topology: which cuts disconnect a surface, and how many cuts can we make before it falls in two pieces? For example, the centerline of the cylinder disconnects it, while a cut from one edge to the other does not. However, a cut from one point to another on the same edge does disconnect the cylinder. The centerline of the Möbius strip does *not* disconnect it.

A cut amounts to a 1-simplex. We will see that cuts that disconnect amount to relative boundaries.

The cylinder: let A be the border of the cylinder, thus A is the disjoint union of two circles, say $A = A_1 \cup A_2$. Although $H_2(\text{cyl}) = 0$, we have $H_2(\text{cyl}, A) = \mathbb{Z}$, since we can put together two triangles to form a 2-chain (call it s) whose boundary lies in A. If a_1 and a_2 are 1-cycles forming the two circles A_1 and A_2, then (with suitable orientations) $\partial s = a_1 - a_2$, and

s generates $H_2(\text{cyl}, A)$. Look next at part of the long exact sequence:

$$\begin{array}{ccccccccc} H_2(\text{cyl}) & \to & H_2(\text{cyl}, A) & \to & H_1(A) & \to & H_1(\text{cyl}) & \to & H_1(\text{cyl}, A) \\ 0 & \to & \mathbb{Z} & \to & \mathbb{Z} \oplus \mathbb{Z} & \to & \mathbb{Z} & \to & \mathbb{Z} \end{array}$$

We already have generators s for $H_2(\text{cyl}, A)$ and a_1, a_2 for $H_1(A)$. Let c be a generator for $H_1(\text{cyl})$. Then:

$$\begin{aligned} \partial(ms) &= ma_1 - ma_2 \\ \partial(ma_1 + na_2) &= (m+n)c \\ \partial c &= 0 \end{aligned}$$

allowing us to check exactness at each spot.

Look next at $H_1(\text{cyl}, A)$. This is also $\mathbb{Z}$, with a generator (call it t) given by a 1-simplex going from a point $p_2 \in A_2$ to a point $p_1 \in A_1$. We have $\partial t = p_1 - p_2$. We look at the next part of the long exact sequence:

$$\begin{array}{ccccccc} H_1(\text{cyl}) & \to & H_1(\text{cyl}, A) & \to & \tilde{H}_0(A) & \to & \tilde{H}_0(\text{cyl}) \\ \mathbb{Z} & \to & \mathbb{Z} & \to & \mathbb{Z} & \to & 0 \end{array}$$

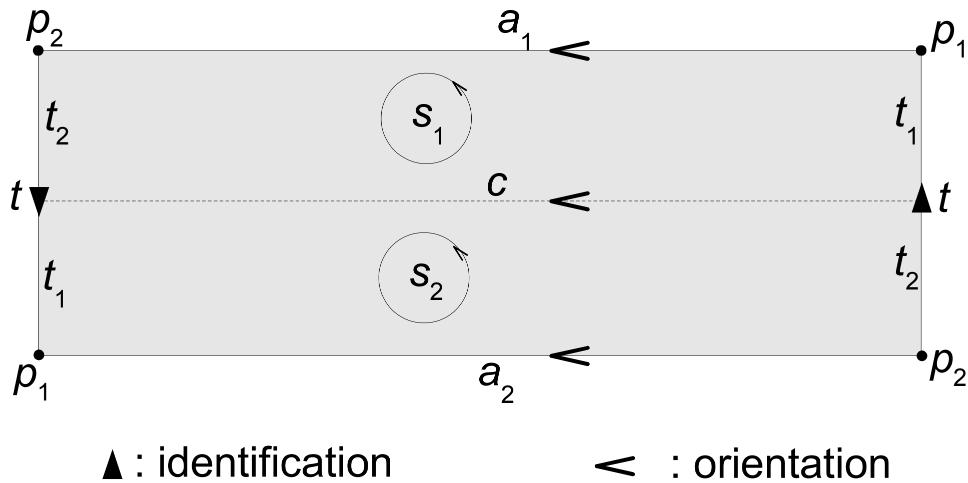
Now we have

$$\begin{aligned} \partial(mt) &= mp_1 - mp_2 \\ \partial(mp_1 - mp_2) &= 0 \end{aligned}$$

again allowing us to check exactness. (Because we are working with the reduced homology group $\tilde{H}_0(A)$, our generic cycle is $mp_1 - mp_2$ and not $mp_1 + np_2$.)

Note that t is a non-disconnecting cut. Also t is not a relative boundary. If we make two edge-to-edge cuts, say t and t', then we do disconnect the cylinder, and indeed $t - t'$ is a relative boundary, thus t and t' are homologous

Figure 2: Möbius Strip



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in the relative group $C_1(\text{cyl}, A)$. The equation $H_1(\text{cyl}, A) = \mathbb{Z}$ captures these facts. Note that the centerline does disconnect and is a relative boundary; correspondingly, the homomorphism $H_1(\text{cyl}) \to H_1(\text{cyl}, A)$ is trivial.

The Möbius strip: Let A be the border of the Möbius strip. A is topologically a circle. As the chain situation is more complicated, we refer to the identified rectangle diagram for the Möbius strip (fig. 2).

The four edges of the rectangle are a_1, a_2, and the identified edges both labelled t. The cycle $a = a1 + a_2$ represents the circle A. The edge t is split into two halves t_1 and t_2. There are two labelled points p_1, p_2, both on A; t goes from p_2 to p_1. A pair of simplices (not shown) combine to make up the rectangular 2-chain s_1, the top half of the diagram; likewise the 2-chain s_2 represents the bottom half. The 2-chain $s = s_1 + s_2$ thus covers the whole Möbius strip. The filled arrowheads indicate identification, the thin arrowheads just orientation.

Thus we have:

$$\begin{aligned}
\partial s_1 &= t_1 + a_1 + t_2 - c = t + a_1 - c \\
\partial s_2 &= t_2 + c + t_1 + t_1 - a_2 = t - a_2 + c \\
\partial s &= 2t + a_1 - a_2 \\
\partial(s_1 - s_2) &= a_1 + a_2 - 2c = a - 2c \\
\partial t &= p_1 - p_2
\end{aligned}$$

This time $H_2(\text{Möb}, A) = 0$ instead of $\mathbb{Z}$ — notice that neither s nor $s_1 - s_2$ has its boundary in A. The cycle a generates $H_1(A)$. The cycle c generates $H_1(\text{Möb})$. The chain t is a relative cycle in $C_1(\text{Möb}, A)$, and generates $H_1(\text{Möb}, A)$.

As before, we want to examine part of the long exact sequence:

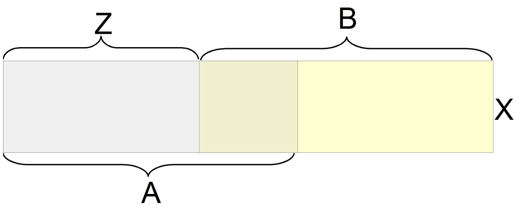
$$\begin{array}{ccccccccc}
H_2(\text{Möb}, A) & \to & H_1(A) & \to & H_1(\text{Möb}) & \to & H_1(\text{Möb}, A) & \to & \tilde{H}_0(A) \\
0 & \to & \mathbb{Z} & \to & \mathbb{Z} & \to & \mathbb{Z}_2 & \to & 0
\end{array}$$

The equation $\partial(s_1 - s_2) = a - 2c$ shows that a is homologous to $2c$, and so the map $i_* : H_1(A) \to H_1(\text{Möb})$ sends $\mathbb{Z}$ to $2\mathbb{Z}$. Exactness then implies that $H_1(\text{Möb}, A) = \mathbb{Z}_2$. But we can also see this from $\partial s = 2t + a_1 - a_2$. This tells us that $2t$ is a relative boundary, that is, it is a boundary in $C_1(\text{Möb}, A)$. Since t generates $H_1(\text{Möb}, A)$, it follows that this group is $\mathbb{Z}_2$.

To conclude, we rephrase our Möbius facts in terms of cuts. The centerline c is not a relative boundary, and it does not disconnect the strip. But: make a cut slightly to one side of the centerline, and follow it twice around the strip until it joins back to itself. This cut is homotopic (hence homologous) to $2c$, and does disconnect.

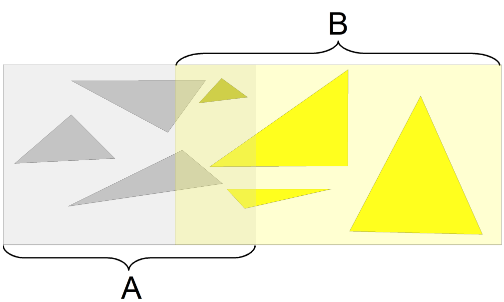
2 §2.1: Excision

p.119: Excision asserts that $H_n(X, A) \approx H_n(X - Z, A - Z)$, provided $\text{cl}(Z) \subset \text{int}(A)$. An equivalent formulation, obtained by letting $B = X - Z$: if $X = A \cup B = \text{int}(A) \cup \text{int}(B)$, then $H_n(A \cup B) \approx H_n(B, A \cap B)$. Thus:

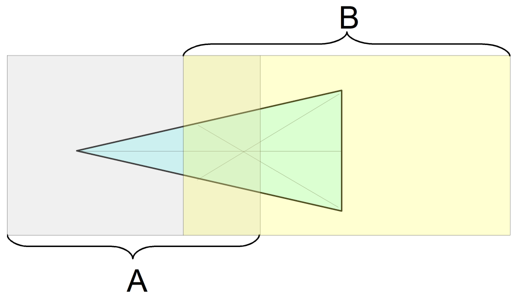


(Note the similarity of this second formulation with Noether's third isomorphism theorem.)

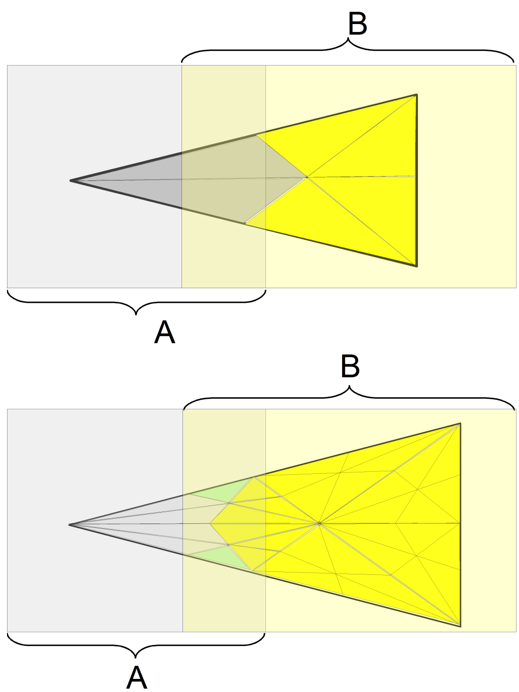
Intuition: stuff in A gets ignored in $H_n(X, A)$, so it doesn't matter if we cut part of A out. This works well for chains composed of "small" simplicies, which all belong either to A or to B. Notation: $C_n(A+B)$ consists of chains made up of such simplicies. The diagram below shows a bunch of "small" simplicies.



But if a chain contains "large" simplicies, which don't lie entirely in A or B, then we have a problem. Solution: use barycentric subdivision to replace the "large" simplicies with "small" simplicies:



It may be necessary to repeat barycentric subdivision several times to obtain "small enough" vertices, as illustrated in this diagram:



To convert this into a formal proof, we turn to the notion of chain homotopy. Recall that a chain homotopy implies equal homology:

$$f - g = \partial D + D\partial \Rightarrow f_* = g_*$$
$$\text{where } f_*, g_* : H(A) \to H(B)$$

$$\begin{tikzcd}
\cdots A_{n+1} \arrow[r, "\partial"] \arrow[d, bend right, "f"'] \arrow[d, bend left, "g"] & A_n \arrow[r, "\partial"] \arrow[dl, "D"'] \arrow[d, bend right, "f"'] \arrow[d, bend left, "g"] & A_{n-1} \cdots \arrow[dl, "D"'] \arrow[d, bend right, "f"'] \arrow[d, bend left, "g"] \\
\cdots B_{n+1} \arrow[r, "\partial"] & B_n \arrow[r, "\partial"] & B_{n-1} \cdots
\end{tikzcd}$$

The goal is to find two chain maps and a chain homotopy: inclusion $\iota : C_n(A+B) \hookrightarrow C_n(A \cup B)$, barycentric subdivision $\rho : C_n(A \cup B) \to C_n(A+B)$, and chain homotopy D. These need to satisfy three equations:

1. $\rho\iota = 1$;
2. $1 - \iota\rho = \partial D + D\partial$; hence
3. $\iota_*\rho_* = 1$.

With these equations, we can construct this commutative diagram:

$$\begin{tikzcd}
C_n(B) \arrow[r] \arrow[d] & C_n(A+B) \arrow[r, shift left, "\iota"] \arrow[d] & C_n(A \cup B) \arrow[l, shift left, "\rho"] \arrow[d] \\
\frac{C_n(B)}{C_n(A \cap B)} \arrow[r, "\approx"] \arrow[d] & \frac{C_n(A+B)}{C_n(A)} \arrow[r, shift left, "\iota"] \arrow[d] & \frac{C_n(A \cup B)}{C_n(A)} \arrow[l, shift left, "\rho"] \arrow[d] \\
H_n(B, A \cap B) \arrow[r, "\approx"] & H_n(A+B, A) \arrow[r, shift left, "\iota"] \arrow[r, phantom, "\approx"] & H_n(A \cup B, A) \arrow[l, shift left, "\rho"]
\end{tikzcd}$$

The bottom line gives excision. A straightforward argument (p.124) shows that the composition $C_n(B) \hookrightarrow C_n(A+B) \to C_n(A+B)/C_n(A)$ is onto with kernel $C_n(A\cap B)$. This establishes the isomorphisms on the left of the diagram.

Here I collect all the equations occurring in Hatcher's proof of excision:

$$
\begin{aligned}
b([w_0,\cdots,w_n]) &= [b,w_0,\cdots,w_n] \quad (b: LC_n(Y)\to LC_{n+1}(Y)) &\quad (3)\\
\partial b([w_0,\cdots,w_n]) &= [w_0,\cdots,w_n] - b(\partial[w_0,\cdots,w_n]) &\quad (4)\\
\partial b(\alpha) &= \alpha - b(\partial\alpha) &\quad (5)\\
\partial b + b\partial &= \mathbf{1} &\quad (6)\\
S(\lambda) &= b_\lambda(S\partial\lambda) \quad (S: LC_n(Y)\to LC_n(Y)) &\quad (7)\\
S([w_0]) &= [w_0] &\quad (8)\\
\partial S &= S\partial &\quad (9)\\
T: LC_n(Y) &\to LC_{n+1}(Y) &\quad (10)\\
T\lambda &= \begin{cases} [w_0w_0] & (n=0, \lambda=[w_0]) \\ b_\lambda(\lambda - T\partial\lambda) & (n>0) \end{cases} &\quad (11)\\
\partial T + T\partial &= \mathbf{1} - S &\quad (12)\\
S\sigma &= \sigma_\# S\Delta^n \quad (S: C_n(X)\to C_n(X)) &\quad (13)
\end{aligned}
$$

$$T\sigma = \sigma_{\#} T\Delta^n \quad (T : C_n(X) \to C_{n+1}(X)) \tag{14}$$

$$D_m = \sum_{i=0}^{m-1} TS^i \quad (D_m : C_n(X) \to C_{n+1}(X)) \tag{15}$$

$$\partial D_m + D_m \partial = \mathbf{1} - S^m \tag{16}$$

$$m(\sigma) = \text{smallest } m \text{ such that } S^m\sigma \subset C_n^{\mathcal{U}}(X) \tag{17}$$

$$D\sigma = D_{m(\sigma)}\sigma \tag{18}$$

$$\partial D\sigma + D\partial\sigma = \sigma - [S^{m(\sigma)}\sigma + D_{m(\sigma)}(\partial\sigma) - D(\partial\sigma)] \tag{19}$$

$$\rho(\sigma) = S^{m(\sigma)}\sigma + D_{m(\sigma)}(\partial\sigma) - D(\partial\sigma) \tag{20}$$

$$\partial D\sigma + D\partial\sigma = \sigma - \rho(\sigma) \tag{21}$$

$$\rho : C_n(X) \to C_n^{\mathcal{U}}(X) \tag{22}$$

$$\partial D + D\partial = \mathbf{1} - \iota\rho \tag{23}$$

$$\iota : C_n^{\mathcal{U}}(X) \hookrightarrow C_n(X) \quad \text{(inclusion map)} \tag{24}$$

Proof of (9), $\partial S = S\partial$:

$$\begin{aligned}
\partial S[w_0] &= \partial[w_0] = 0 = S\partial[w_0] \\
\partial S\lambda &= \partial(b_\lambda(S\partial\lambda)) \\
&= S\partial\lambda - b_\lambda(\partial S\partial\lambda) \quad \text{since } \partial b_\lambda + b_\lambda\partial = \mathbf{1} \\
&= S\partial\lambda - b_\lambda(S\partial\partial\lambda) \quad \text{by induction on } n \\
&= S\partial\lambda \quad \text{since } \partial\partial = 0
\end{aligned}$$

Proof of (12), $\partial T + T\partial = \mathbf{1} - S$ (i.e., $\partial T = \mathbf{1} - T\partial$, i.e., $\mathbf{1} - \partial T = S + T\partial$):

$$\begin{aligned}
\partial T[w_0] &= \partial[w_0 w_0] = 0 = [w_0] - S[w_0] - T\partial[w_0 w_0] \\
\partial T\lambda &= \partial(b_\lambda(\lambda - T\partial\lambda)) \\
&= \lambda - T\partial\lambda - b_\lambda(\partial(\lambda - T\partial\lambda)) \quad \text{since } \partial b_\lambda = \mathbf{1} - b_\lambda\partial \\
&= \lambda - T\partial\lambda - b_\lambda(\partial\lambda - \partial T\partial\lambda) \\
&= \lambda - T\partial\lambda - b_\lambda((\mathbf{1} - \partial T)\partial\lambda) \\
&= \lambda - T\partial\lambda - b_\lambda((S + T\partial)\partial\lambda) \quad \text{by induction on } n \\
&= \lambda - T\partial\lambda - b_\lambda(S\partial\lambda + T\partial\partial\lambda) \\
&= \lambda - T\partial\lambda - S\lambda \quad \text{since } \partial\partial = 0 \text{ and } S\lambda = b_\lambda(S\partial\lambda)
\end{aligned}$$

Proof that S in (13) is a chain map, $\partial S = S\partial$:

$$\begin{aligned}
\partial S\sigma &= \partial\sigma_\# S\Delta^n = \sigma_\#\partial S\Delta^n = \sigma_\# S\partial\Delta^n \\
&= \sigma_\# S\left(\sum_i (-1)^i \Delta^n_i\right) \\
&= \sum_i (-1)^i \sigma_\# S\Delta^n_i \\
&= \sum_i (-1)^i S(\sigma|\Delta^n_i) \\
&= S\left(\sum_i (-1)^i \sigma|\Delta^n_i\right) = S(\partial\sigma)
\end{aligned}$$

Proof that T in (14) is a chain homotopy, $\partial T = \mathbf{1} - S - T\partial$:

$$\begin{aligned}
\partial T\sigma &= \partial\sigma_\# T\Delta^n \\
&= \sigma_\#\partial T\Delta^n \\
&= \sigma_\#(\Delta^n - S\Delta^n - T\partial\Delta^n) \\
&= \sigma - S\sigma - \sigma_\# T(\partial\sigma) \\
&= \sigma - S\sigma - T\partial\sigma
\end{aligned}$$

Proof that D_m is a chain homotopy (16):

$$\begin{aligned}
\partial D_m + D_m\partial &= \sum_{i=0}^{m-1}(\partial T S^i + T S^i \partial) \\
&= \sum_{i=0}^{m-1}(\partial T S^i + T\partial S^i) \\
&= \sum_{i=0}^{m-1}(\partial T + T\partial) S^i \\
&= \sum_{i=0}^{m-1}(\mathbf{1} - S) S^i \\
&= \sum_{i=0}^{m-1}(S^i - S^{i+1}) = \mathbf{1} - S^m
\end{aligned}$$

In (17), to show there is an m with $S^m\sigma \subset C_n^{\mathcal{U}}(X)$, we use σ^{-1} to pull back the open cover $\mathcal{U}$ to get an open cover of the compact set Δ^n. By the Lebesgue covering lemma, there is an ϵ such that any subset of Δ^n of diameter less than ϵ lies in one of the sets of the cover. Iterating S will produce sets of arbitrarily small diameter.

The reason for the complicated definition of ρ, (20), is that $S^{m(\sigma)}$ does not determine a chain map. That's because $m(\sigma)$ can be larger than $m(\sigma_i)$ for a face σ_i of σ. Note that $m(\sigma)$ cannot be smaller than $m(\sigma_i)$.

Now compare $D_{m(\sigma)}(\partial\sigma)$ with $D\partial\sigma$. $D_{m(\sigma)}(\partial\sigma)$ may go a few iterations past $D\partial\sigma$ on some simplicies in $\partial\sigma$, but this can happen only after we're already in $C_{n-1}^{\mathcal{U}}(X)$. It follows that $D_{m(\sigma)}(\partial\sigma) - D(\partial\sigma)$ lies in $C_{n-1}^{\mathcal{U}}(X)$, and so $\rho(\sigma)$ does too (22).

Finally, ρ is a chain map. Proof: note first that by (21)

$$\rho(\partial\sigma) = \partial\sigma - \partial D\partial\sigma - D\partial\partial\sigma = \partial\sigma - \partial D\partial\sigma$$

But also:

$$\partial\rho(\sigma) = \partial(\sigma - \partial D\sigma - D\partial\sigma) = \partial\sigma - \partial D\partial\sigma$$

3 §2.2: Mapping Telescope

p.138: Lemma 2.34(c) (p.137) states

> The inclusion $i : X^n \hookrightarrow X$ induces an isomorphism $i_* : H_k(X^n) \to H_k(X)$ if $k < n$.

Intuitively, cells of dimension higher than n cannot affect homology in dimensions lower than n. Note that cells of dimension n *could* affect $H_{n-1}(X)$ by turning $(n-1)$-cycles into boundaries.

Hatcher first proves 2.34(c) for the case where X is finite-dimensional. To extend this to general X, he first sketches a compactness argument, and then describes the "mapping telescope" construction. He shows that 2.34(c) follows from this assertion:

> (*) $\tilde{H}_k(X) = 0$ for $k \leq n$ if the n-skeleton of X is a point.

Note: $k \leq n$, not $k < n$. Intuitively this is clear, since cells of dimension $n+1$ can only trivialize n-cycles. The $k = n$ case is needed to show 2.34(c), since he uses this part of the long exact sequence: $H_{k+1}(X, X^n) \to H_k(X^n) \to H_k(X) \to H_k(X, X^n)$.

Hatcher then writes: "When X is finite-dimensional, (*) is immediate from the finite-dimensional case of (c) which we have already shown." Thus: obviously $\tilde{H}_k(X^n) = 0$ if $X^n = \{\text{point}\}$, and 2.34(c) tells us that $\tilde{H}_k(X^n) \approx \tilde{H}_k(X)$, so $\tilde{H}_k(X) = 0$. But this works only for $k < n$.

To fix this, beef up the statement of 2.34(c):

> The inclusion $i : X^n \hookrightarrow X$ induces an isomorphism $i_* : H_k(X^n) \to H_k(X)$ if $k < n$, and induces an onto map $i_* : H_n(X^n) \to H_n(X)$.

This handles the $k = n$ case of (*): since $\tilde{H}_n(X^n) = 0$ and $\tilde{H}_n(X^n) \to \tilde{H}_n(X)$ is onto, $\tilde{H}_n(X)$ must be zero also. To prove the supplement, look at this part of the long exact sequence: $H_n(X^n) \to H_n(X^{n+1}) \to H_{n-1}(X^{n+1}, X^n) = 0$, the "$= 0$" coming from 2.34(a). Then we have $H_n(X^{n+1}) \approx H_n(X^{n+2}) \approx \cdots \approx H_n(X^{n+m})$ for all $m \geq 1$, just as in Hatcher, so we are done.

4 §2.2: Cellular Homology

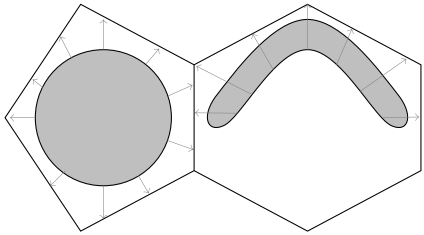
p.139: Look at the commutative diagram on p.139. The groups $H_{n+1}(X^{n+1}, X^n)$, $H_n(X^n, X^{n-1})$, and $H_{n-1}(X^{n-1}, X^{n-2})$ along the "spine" have a direct visual interpretation, thanks to Lemma 2.34(a) (p.137): $H_n(X^n, X^{n-1})$ is free abelian on the n-cells e^n_α. Likewise for the other two groups *mutatis mutandis.*

The map $d_n = j_{n-1}\partial_n$ also lends itself to visualization, with a little more work. We look at the case $n = 3$. Imagine X^3 as a foam of tiny bubbles all pressed together. Each bubble is a 3-cell, its interior made of air and its soap-film boundary contained in the 2-skeleton X^2. This boundary is a singular 2-cycle in $H_2(X^2)$. You should picture it as a featureless sphere, not as tiled into 2-cells: $H_2(X^2)$ regards X^2 as a topological space, whose decomposition into 2-cells carries no special significance.

Passing to the relative group $H_2(X^2, X^1)$ changes matters: the 2-cells of X^2 correspond to basis elements of $H_2(X^2, X^1)$. In other words, we can picture j_2 as a "tiling" homomorphism. The composition $j_2\partial_3$ thus maps a basis element of $H_3(X^3, X^2)$ to a sum of basis elements of $H_2(X^2, X^1)$.

Fig.3 illustrates the same kind of tiling for $n = 2$. The shaded areas are

Figure 3: Cellular Homology



The shaded areas are 2-cells, attached to the 1-skeleton via radial projection.

2-cells of the 2-skeleton, with the attaching maps φ_α indicated by radial projection. The boundary of the left-hand 2-cell is tiled into five 1-cells by j_1, and d_2 of the grey disk is a sum of five elements of $H_1(X^1, X^0)$.

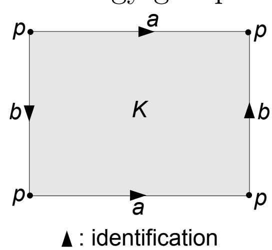
The right-hand side shows how loosey-goosey CW-complexes are, compared to Δ-complexes. In a Δ-complex, the boundary of an n-simplex is composed of a bunch of $(n-1)$-simplices, each a building block of the $(n-1)$-skeleton. For a CW-complex, the boundary of an n-cell is attached (via φ_α) to the $(n-1)$-skeleton, but the *only* requirement is that φ_α map into X^{n-1}.

The cellular boundary formula (p.140–141) deals with this. Consider the map $\Delta_{\alpha\beta}$, defined as the composition $q_\beta q \varphi_\alpha$ on p.141. Here φ_α attaches the cell e^n_α to the $(n-1)$-skeleton. Then q collapses the $(n-2)$-skeleton to a point, so the image of ∂e^n_α under $q\varphi_\alpha$ is distributed over $(n-1)$-spheres making up X^{n-1}/X^{n-2}; e^{n-1}_β corresponds to one of them. Finally q_β collapses everything but e^{n-1}_β to a point. Net result: $\Delta_{\alpha\beta} : \partial D^n_\alpha \to S^{n-1}_\beta$, a map from one $(n-1)$-sphere to another, of degree $d_{\alpha\beta}$.

Picture this for the right-hand side of fig.3. The right and left vertical edges of the hexagon (e^1_β's) collapse to circles, and the boundary of the boomerang-shaped 2-cell maps to loops, each covering only a part of their respective circles. Thus $d_{\alpha\beta} = 0$ for these β's. The two top edges of the hexagon also collapse to circles, with the boundary of the boomerang mapping to two oppositely-oriented loops for each circle. Thus $d_{\alpha\beta} = 0$ here also.

5 Homology of the Klein Bottle

p.141: Examples 2.36 and 2.37 compute the homology of the surfaces of genus g using the cellular technique. The Klein bottle K is the nonorientable surface of genus 1 (see p.51), so this tells us its homology groups: $H_2 = 0$, $H_1 = \mathbb{Z} \oplus \mathbb{Z}_2$, $H_0 = \mathbb{Z}$. However, it's an easy exercise to compute the homology of K from the usual diagram. Let W_2, W_1, W_0 be the cellular chain groups $H_i(X^i, X^{i-1})$. We let K stand for the generator of W_2, a and b for the generators of W_1, and p for the generator of W_0. Then we have the chain complex



$$0 \longrightarrow \overset{W_2}{\mathbb{Z}} \xrightarrow{K \mapsto 2b} \overset{W_1}{\mathbb{Z} \oplus \mathbb{Z}} \xrightarrow{0} \overset{W_0}{\mathbb{Z}} \longrightarrow 0 \tag{25}$$

So the kernel of the map $d_2 : W_2 \to W_1$ is 0; the image of d_2 is $2\mathbb{Z}$, lying in the second summand of $\mathbb{Z} \oplus \mathbb{Z}$. The homology groups follow immediately.

We will revisit this when we look at cohomology.

6 Example 2.43: Lens Spaces

p.144: The diagram on p.145 illustrates the special case $n = 2$, $(m, l) = (7, 2)$. Hatcher is using stereographic projection to identify S^3 with $\mathbb{R}^3$ plus the point at infinity. The vertical axis represents the circle C. C

is subdivided into 7 segments (equal in S^3, not in the stereographically projected image); as Hatcher says, the subdividing vertices are joined to the sphere S^{2n-3} via arcs of great circles to produce the balls B_j^{2n-2}. In the diagram, the sphere S^{2n-3} is just the horizontal circle. Since stereographic projection preserves circles and spheres[1], the balls B_j^{2n-2} are spherical caps in $\mathbb{R}^3$. Two are shown in the diagram, with a B_j^{2n-1} represented as the lens-shaped region between them.

7 §2.A: Homology and the Fundamental Group

p.166: Hatcher proves that the abelianized fundamental group, $\pi_1(X)_{\text{ab}} = \pi_1(X)/[\pi_1(X), \pi_1(X)]$, is isomorphic to $H_1(X)$. His proof in an earlier version was slightly different. We explain the differences and clarify a couple of small points.

The key to the argument are two homomorphisms: $\alpha : \pi_1(X) \to \pi_1(X)_{\text{ab}}$ and $\eta : \pi_1(X) \to H_1(X)$. (Hatcher does not give explicit names to these homomorphisms.) The map α is the usual group-theoretical abelianization, while η springs from the idea of regarding a loop in X as a singular 1-cycle in the obvious way. Hatcher's description of η is quite clear, so I won't elaborate.

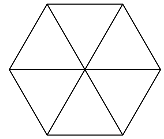
The central part of the proof is showing that if $\eta[f] = 0$, then $\alpha[f] = 0$, where f is a loop in X. (The converse is obvious. Here, [ ] stands for homotopy class.)

If $\eta[f] = 0$, that means that $f \sim 0$, i.e., $f = \partial \sum_i n_i\sigma_i$. Hatcher now argues that the singular 1-simplicies occuring in $\partial \sum_i n_i\sigma_i$ must cancel in pairs, leaving only f. This is correct, but visually confusing: it's natural to picture the statement $f \sim 0$ as a bunch of little triangles tiling a polygon,

[1] counting lines as circles passing through infinity, likewise for planes and spheres

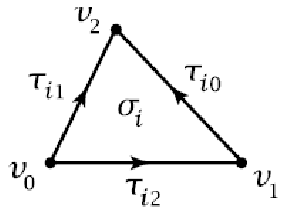
with f being the boundary of the polygon.

For example, consider a hexagon triangulated by its three diameters. Its boundary is the sum of six terms, one for each line segment, with the interior line segments cancelling in pairs.



But we want to have only one singular 1-simplex left after all the cancellations: the loop around the entire hexagon. The explanation is that f is homologous to that sum of six terms; the 2-chain establishing this is degenerate. This becomes clear from looking at (iii) on p.166.

Let $f = \partial \sum_i n_i \sigma_i = \sum_i n_i \partial \sigma_i$. The boundary of each σ_i is made up of three pieces, $\sigma_i = \tau_{i0} - \tau_{i1} + \tau_{i2}$ (see Hatcher's diagram on p.167, reproduced at right). There is an obvious sense in which $\tau_{i0} \cdot \tau_{i1}^{-1} \cdot \tau_{i2}$ is homotopic to 1; thus we *almost* have the equation

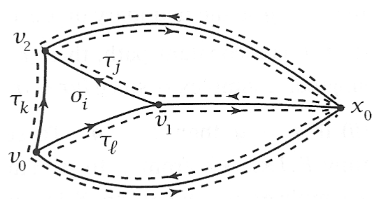


$$f \sim \prod_i \tau_{i0} \cdot \tau_{i1}^{-1} \cdot \tau_{i2} \sim 1 \qquad \text{(NOT QUITE!!!)} \tag{26}$$

One problem is that the τ_{ij} are not loops! In Hatcher's original proof, he dealt with this via a standard technique: let x_0 be the basepoint of $\pi_1(X)$, let $\{v_a\}$ be the vertices occurring among all the τ_{ij}, and for each a choose an arc δ_a from x_0 to v_a. Then if τ_{ij} goes from v_a to v_b, we have a loop $\delta_a \tau_{ij} \delta_b^{-1}$, based at x_0, and thus its homotopy class belongs to $\pi_1(X)$. Writing $\tilde{\tau}_{ij}$ for this loop, we have an equation

$$\prod_i [\tilde{\tau}_{i0}] \cdot [\tilde{\tau}_{i1}]^{-1} \cdot [\tilde{\tau}_{i2}] = 1 \qquad (\text{in } \pi_1(X)) \tag{27}$$

Hatcher even drew a diagram to illustrate this (at right). (In the earlier version, Hatcher used one subscript for the τ's instead of two. With that notation, the relation reads $\tilde{\tau}_j \cdot \tilde{\tau}_k^{-1} \cdot \tilde{\tau}_\ell \sim 1$.)



The other issue is that we don't necessarily have $f \sim \prod_i \tilde{\tau}_{i0} \cdot \tilde{\tau}_{i1}^{-1} \cdot \tilde{\tau}_{i2}$, even with these new loopy $\tilde{\tau}_{ij}$'s. A pair cancellation can't take place if a $[\tilde{\tau}_{ij}]$ and its inverse are not adjacent in the product (27).

However, when when we pass to $\pi_1(X)_{\mathrm{ab}}$, we're golden. As noted above, from $\eta[f] = 0$ we get

$$f = \sum_i \tau_{i0} - \tau_{i1} + \tau_{i2} \qquad (\text{in } C_1(X)) \tag{28}$$

This is an equation between 1-chains. Now we can still form the product on the left-hand-side of (27) in $\pi_1(X)$, and abelianize it (i.e., apply α) to get:

$$\sum_i \alpha[\tilde{\tau}_{i0}] - \alpha[\tilde{\tau}_{i1}] + \alpha[\tilde{\tau}_{i2}] = 0 \qquad (\text{in } \pi_1(X)_{\mathrm{ab}}) \tag{29}$$

As noted before, the only way for (28) to hold is for the τ's to cancel in pairs except for one, that one being equal to f. The exact same computation must hold in $\pi_1(X)_{\mathrm{ab}}$, giving

$$\alpha[f] = \sum_i \alpha[\tilde{\tau}_{i0}] - \alpha[\tilde{\tau}_{i1}] + \alpha[\tilde{\tau}_{i2}] \qquad (\text{in } \pi_1(X)_{\mathrm{ab}}) \tag{30}$$

Putting (29) and (30) together, we have $\alpha[f] = 0$ in $\pi_1(X)_{\mathrm{ab}}$. End of proof.

In Hatcher's newer version, he throws out all the δ's, and instead constructs a Δ-complex K, with a map $\sigma : K \to X$. He deforms σ so that all the vertices of K map to the basepoint x_0. Now if you visualize this deformation, you'll see that it amounts to having the images of the vertices trace out arcs connecting them to x_0 — basically, the δ's in disguise (or rather their inverses).

I prefer Hatcher's original proof. On the other hand, he uses the Δ-complex K on p.168 in an illuminating discussion of the geometry of the isomorphism.

8 Proposition 2B.1: Embeddings of spheres in spheres

p.170: As a corollary to Proposition 2B.1, Hatcher concludes that any embedding of S^n into S^n is surjective. He uses the equation $\tilde{H}_{-1}(\varnothing) = \mathbb{Z}$. Delightful as it is to contemplate the reduced -1-dimensional homology group of the empty set, note that the embedding result is a special case of Corollary 2B.4 (p.172), proved independently.

Let's look at $\tilde{H}_{-1}(\varnothing) = \mathbb{Z}$. For any space X, the reduced groups come from the chain complex

$$\cdots \xrightarrow{\partial} \overset{\dim=0}{C_0(X)} \xrightarrow{\varepsilon} \overset{\dim=-1}{\mathbb{Z}} \longrightarrow 0$$

For $X = \varnothing$, this becomes

$$\cdots \xrightarrow{\partial} \overset{\dim=0}{0} \xrightarrow{\varepsilon} \overset{\dim=-1}{\mathbb{Z}} \longrightarrow 0$$

so in dimension -1, $\tilde{H}_{-1}(\varnothing) = \mathbb{Z}$, while in all other dimensions $\tilde{H}_n(\varnothing) = 0$. On the other hand, if $X \neq \varnothing$, then $C_0(X) \neq 0$ and the image of ε is all of $\mathbb{Z}$, so $\tilde{H}_{-1}(X) = 0$.

Let's see how this fits into the reduced Mayer-Vietoris sequence (p.150). The reduced long exact sequence derives from a short exact sequence of augmented chain complexes:

$$
\begin{tikzcd}
 & \overset{\dim=0}{0} \arrow[d] & \overset{\dim=-1}{0} \arrow[d] & \\
\cdots \arrow[r] & C_0(A\cap B) \arrow[r, "\varepsilon"] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d] & 0 \\
\cdots \arrow[r] & C_0(A)\oplus C_0(B) \arrow[r, "\varepsilon\oplus\varepsilon"] \arrow[d] & \mathbb{Z}\oplus\mathbb{Z} \arrow[r] \arrow[d] & 0 \\
\cdots \arrow[r] & C_0(A+B) \arrow[r, "\varepsilon"] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 &
\end{tikzcd}
$$

(I'm writing the exact sequences vertically, whereas p.150 has them horizontal.) If A and B are nonempty but $A\cap B=\varnothing$, then we get:

$$
\begin{tikzcd}
\cdots \arrow[r] & \overset{\dim=0}{0} \arrow[r, "\varepsilon"] \arrow[d, "\varphi"] & \overset{\dim=-1}{\mathbb{Z}} \arrow[r] \arrow[d, "\varphi"] & 0 \\
\cdots \arrow[r] & C_0(A)\oplus C_0(B) \arrow[r, "\varepsilon\oplus\varepsilon"] \arrow[d, "\psi"] & \mathbb{Z}\oplus\mathbb{Z} \arrow[r] \arrow[d, "\psi"] & 0 \\
\cdots \arrow[r] & C_0(A+B) \arrow[r, "\varepsilon"] & \mathbb{Z} \arrow[r] & 0
\end{tikzcd}
$$

and the 0 and -1 dimensional parts of the reduced Mayer-Vietoris sequence read

$$\cdots\to 0\to \tilde{H}_0(A)\oplus\tilde{H}_0(B)\to\tilde{H}_0(A\cup B)\to\mathbb{Z}\to 0\to 0$$

The ranks check out: if A has a path components and B has b path components, then $A\cup B$ has $a+b$ path components (disjoint union). So

$\tilde{H}_0(A) \oplus \tilde{H}_0(B) \approx \mathbb{Z}^{a+b-2}$ and $\tilde{H}_0(A \cup B) \approx \mathbb{Z}^{a+b-1}$.

On the other hand, if $A \cap B \neq \varnothing$, then the sequence ends

$$\cdots \to \tilde{H}_0(A \cap B) \to \tilde{H}_0(A) \oplus \tilde{H}_0(B) \to \tilde{H}_0(A \cup B) \to 0$$

and $A \cup B$ will always have fewer path components than in the disjoint union case.

In the proof of Proposition 2B.1(b), an induction using the reduced Mayer-Vietoris sequence shows that $\tilde{H}_i(S^n - h(S^{k-1})) \approx \tilde{H}_{i-1}(S^n - h(S^k))$. The argument retains its validity even when $k = n$ and $i = 0$ — provided we use the reduced sequences just described. So $\tilde{H}_{-1}(S^n - h(S^n)) = \tilde{H}_0(S^n - h(S^{n-1})) = \mathbb{Z}$ (using Prop. 2B.1(b) for the latter equality). So we must have $S^n - h(S^n) = \varnothing$, i.e., the embedding h is onto.

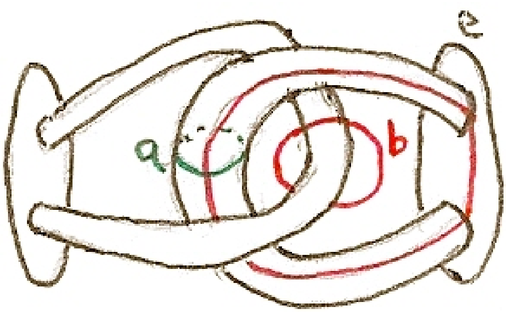
9 Example 2B.2: The Alexander Horned Sphere

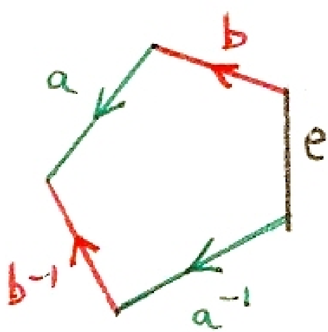
p.170: Following this example gives the visualization muscles quite a work-out. (Cool animations of the Alexander Horned Sphere may be found on Youtube.) Begin with Hatcher's diagram on p.171. The "horned ball" is both the intersection of a decreasing sequence $X_0 \supset X_1 \supset \ldots$ and the union of an increasing sequence $B_0 \subset B_1 \subset \ldots$. A guide to his notation, as it applies to the picture:

- B_0 is the ball. B_1 is B_0 plus the two fat horns. B_2 is B_1 plus the four skinnier horns.
- X_0 is the ball B_0 plus the fat handle; the handle is made up of the horns plus the connecting "dashed" tube; by "tube", I mean the filled-in solid tube. X_1 is X_0 with this tube deleted, including its outer sheath, but *not* including the two interlocking skinnier handles.

- The Y_i's are the complements of the X_i's, so they are open sets.
- The open annulus $A = \overline{Y_0} - Y_0$ is what I just called the outer sheath of dashed tube. So A is represented in the picture by the dashed lines. A is topologically an annulus, not including the two circular edges.
- $Z = Y_1 - Y_0$ is the stuff inside the outer sheath A, not including the skinnier handles, plus the sheath itself.

The following diagram will help to picture the assertion that α is homotopic to the commutator $[\alpha_1, \alpha_2]$. In this diagram, $a = \alpha_1$, $b = \alpha_2$, $e = \alpha$.



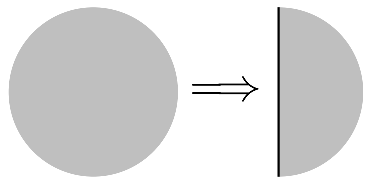


$$e \simeq a^{-1} b^{-1} a b$$

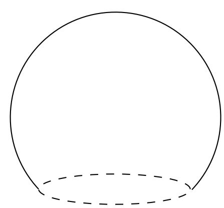
10 §2C: Invariance of Domain

p.172: For further insight into the proof of Theorem 2B.3 (Invariance of Domain), consider a couple of cases of mapping an open disk into $\mathbb{R}^2$, with the image not open.

First, fold the disk over along a diameter and lay it down on the plane, as in the figure at right. The proof invites us to look at $A = h(D^2 - \partial D^2)$ and $B = S^2 - h(D^2)$, where D^2 is the closed disk, h is the mapping, and S^2 is the one-point compactification of $\mathbb{R}^2$. A and B are disjoint, and A is the image of the open disk. Also $A \cup B = S^2 - h(\partial D^2)$, using the disjointness of the images of the open disk and its boundary (for this h). Nonetheless A is not a component or a path component; $A \cup B$ is the single path component of $S^2 - h(\partial D^2)$.



With a different h, we can even make the image of the open disk a closed disk. Represent the open disk as a spherical cap that is more than a hemisphere, as in the figure at right. Let h map it to the plane via vertical projection. The image in the plane is a closed disk. A and B are still disjoint, and this time $S^2 - h(\partial D^2)$ does have two components! However $A \cup B = S^2$ instead of $S^2 - h(\partial D^2)$.



These "counter-examples" throw into relief the proof strategy: when h is an embedding, $A \cup B = S^2 - h(\partial D^2)$ and $S^2 - h(\partial D^2)$ has two path components. We conclude that A and B are the two components of $S^2 - h(\partial D^2)$, and since components are always closed, A must be open in $S^2 - h(\partial D^2)$ and hence in S^2.

11 §2C: Simplicial Approximation

p.177: Three facts about simplicial complexes to bear in mind:

1. Different simplicies have disjoint interiors.
2. If the simplex $[u_1, \dots, u_m]$ belongs to the complex, and we delete some of the u_i's to obtain a sequence $v_1, \dots, v_n$, then $[v_1, \dots, v_n]$ also belongs to the complex.
3. If $\sigma \subset \tau$, then the vertices of σ are a subsequence of the vertices of τ, as in the previous item.

These are used tacitly in the proof of Lemma 2C.2, in the first sentence: $p \in \operatorname{st} v_1 \cap \cdots \cap \operatorname{st} v_n \Leftrightarrow \forall i\, \exists \tau_i (p \in \text{interior}\, \tau_i \text{ and } v_i \in \tau_i) \Leftrightarrow$ (using (1)) $\exists \tau\, \forall i (p \in \text{interior}\, \tau \text{ and } v_i \in \tau)$; now using (2) and (3) it follows that $[v_1, \dots, v_n]$ belongs to the complex, and so $p \in \operatorname{st} v_1 \cap \cdots \cap \operatorname{st} v_n \Leftrightarrow \exists \tau (p \in \text{interior}\, \tau \text{ and vertex set}\, \tau \supset \{v_1, \dots, v_n\})$.

The proof of 2C.1 demands that $v \in \operatorname{st} v$ for any vertex v. Otherwise $\{f^{-1}(\operatorname{st} w)|\ w \text{ is a vertex of } L\}$ might not cover K. So the interior of $[v]$ is $\{v\}$. Hatcher defines the interior of τ as $\tau - \partial\tau$, and as a chain equation $\partial[v] = 0$, so this is (sort of) consistent. The topological interior of $\{v\}$ is empty, though, unless v is an isolated point.

More generally, if σ is a face of another simplex, then the interior of σ is not open: only top-dimensional simplices have open interiors. So the statement that $\operatorname{st} v$ is open (occurring right before Lemma 2C.2) requires some justification. Note first that K is the disjoint union of $\operatorname{int}\sigma$ over all simplices σ. We can divide these simplex interiors into two disjoint classes: those σ's having v as a vertex (call it A) and the rest (call it B). So the space is a disjoint union $\bigcup_{\sigma\in A} \operatorname{int}\sigma \cup \bigcup_{\sigma\in B} \operatorname{int}\sigma$. But the union $\bigcup_{\sigma\in B} \operatorname{int}\sigma$ is in fact a union of *closed* simplices: if σ belongs to B, then any subsimplex of σ also belongs to B, and so the closed simplex $\sigma = \bigcup_{\tau \subseteq \sigma} \operatorname{int} \tau$ is contained in that union. So $\bigcup_{\sigma \in B} \operatorname{int} \sigma$ is a finite union of closed sets (finite by assumption), and so is closed, making $\operatorname{st} v$ open. As noted above, for any simplex σ, $\operatorname{st} \sigma$ is the intersection $\bigcap \operatorname{st} v$ over the vertices of σ, so that's open too.

In a simplicial complex, the vertices of a simplex must be distinct — degenerate simplicies would violate (1) plus (2). But the vertices $g(v_1), \ldots, g(v_n)$ might not all be distinct. So strictly speaking, instead of saying that $[g(v_1), \ldots, g(v_n)]$ is a simplex of L, Hatcher should have said that after eliminating duplicates from $g(v_1), \ldots, g(v_n)$ and possibly reordering, the resulting vertices determine a simplex of L. This is adequate for Hatcher's purposes.

12 §2C: Additivity of Trace

p.181: In the commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "i"] \arrow[d, "\alpha"] & B \arrow[r, "j"] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & A \arrow[r, "i"] & B \arrow[r, "j"] & C \arrow[r] & 0
\end{tikzcd} \tag{31}$$

we have $\operatorname{tr} \beta = \operatorname{tr} \alpha + \operatorname{tr} \gamma$. Here are the details of the argument. First consider the case where A, B, and C are free abelian. Then $B \approx A \oplus C$, so we can choose a basis for B obtained by concatenating a basis for A with a basis for C. The matrix for β takes the block form

$$\left[\begin{array}{ccc|ccc}
\alpha_{11} & \cdots & \alpha_{1k} & \cdot & \cdot & \cdot \\
\vdots & \ddots & \vdots & \cdot & \cdot & \cdot \\
\alpha_{k1} & \cdots & \alpha_{kk} & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \gamma_{11} & \cdots & \gamma_{1l} \\
\cdot & \cdot & \cdot & \vdots & \ddots & \vdots \\
\cdot & \cdot & \cdot & \gamma_{l1} & \cdots & \gamma_{ll}
\end{array}\right]$$

from which the result follows immediately.

To reduce the general case to this case, we divide out by torsion. Let $\overline{A} = A/\text{torsion}\,A$, likewise for $\overline{B}$ and $\overline{C}$. Abusing notation slightly, let τ stand for each the canonical homomorphisms $A \to \overline{A}$, $B \to \overline{B}$, $C \to \overline{C}$. Then we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "i"] \arrow[d, "\tau"] & B \arrow[r, "j"] \arrow[d, "\tau"] & C \arrow[r] \arrow[d, "\tau"] & 0 \\
0 \arrow[r] & \overline{A} \arrow[r, "\bar\imath"] & \overline{B} \arrow[r, "\bar\jmath"] & \overline{C} \arrow[r] & 0
\end{tikzcd} \tag{32}$$

It will make life easier if we assume (without loss of generality) that $i : A \to B$ is inclusion, so $A = \ker j$. Difficulty: the second row of (32) might not be exact at B. Example: let $A = 2\mathbb{Z}$, $B = \mathbb{Z}$, $C = \mathbb{Z}_2$. Then (32) becomes

$$\begin{tikzcd}
0 \arrow[r] & 2\mathbb{Z} \arrow[r, hook, "i"] \arrow[d, "\tau"] & \mathbb{Z} \arrow[r, "j"] \arrow[d, "\tau"] & \mathbb{Z}_2 \arrow[r] \arrow[d, "\tau"] & 0 \\
0 \arrow[r] & 2\mathbb{Z} \arrow[r, hook, "\bar\imath"] & \mathbb{Z} \arrow[r, "\bar\jmath"] & 0 \arrow[r] & 0
\end{tikzcd}$$

The solution is to enlarge A to $A' = \tau^{-1}[\ker \bar\jmath]$. A quick chase around the right-hand rectangle shows that $A \subset A'$. Diagram (31) becomes

$$\begin{tikzcd} 0 \arrow[r] & A' \arrow[r, "i"] \arrow[d, "\alpha'"] & B \arrow[r, "j"] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & 0 \\ 0 \arrow[r] & A' \arrow[r, "i"] & B \arrow[r, "j"] & C \arrow[r] & 0 \end{tikzcd} \tag{33}$$

Here we define α' as the restriction of β to A'; a diagram chase shows that α is the restriction of β to A. The rows of (33) are not exact, but dividing by the torsion gives:

$$\begin{tikzcd} 0 \arrow[r] & \overline{A'} \arrow[r, "\bar\imath"] \arrow[d, "\overline{\alpha'}"] & \overline{B} \arrow[r, "\bar\jmath"] \arrow[d, "\overline{\beta}"] & \overline{C} \arrow[r] \arrow[d, "\overline{\gamma}"] & 0 \\ 0 \arrow[r] & \overline{A'} \arrow[r, "\bar\imath"] & \overline{B} \arrow[r, "\bar\jmath"] & \overline{C} \arrow[r] & 0 \end{tikzcd} \tag{34}$$

Now, (34) is exact at $\overline{B}$ by the definition of A'. It's not hard to check that the rows are exact in the other places as well. So we have $\operatorname{tr}\overline{\beta} = \operatorname{tr}\overline{\alpha'} + \operatorname{tr}\overline{\gamma}$, the free abelian case having been proved. Hatcher defines the trace of a homomorphism by dividing out the torsion, so if we can show that $\operatorname{tr}\overline{\alpha'} = \operatorname{tr}\overline{\alpha}$, we can then conclude that $\operatorname{tr}\beta = \operatorname{tr}\alpha + \operatorname{tr}\gamma$.

First note that for any $x \in A'$, there is a $k \in \mathbb{N}$ such that $kx \in A$. Proof: $x \in A'$ implies $\overline{x} \in \ker \bar\jmath$, i.e., $\bar\jmath\,\overline{x} = 0$, so $jx \in \operatorname{torsion} C$, say $jx = c$ with $kc = 0$ for some $k \in \mathbb{N}$. Then $jkx = kjx = kc = 0$, i.e., $kx \in \ker j = A$. Informally, we can say that $x = (1/k)a$ for $a \in A$.

The simplest way to finish the argument is to tensor with $\mathbb{Q}$. This kills the torsion, and allows us to write $x = (1/k)a$ formally, the latter being an element of $\mathbb{Q}\otimes A$. In other words, $\mathbb{Q}\otimes A' = \mathbb{Q}\otimes A$. Since α is the restriction of α' to A, we must have $\mathbb{Q}\otimes\alpha' = \mathbb{Q}\otimes\alpha$. Finally, $\operatorname{tr}\mathbb{Q}\otimes\alpha = \operatorname{tr}\overline{\alpha}$ (since tensoring with $\mathbb{Q}$ kills torsion), likewise for α', and we are done.

13 §2C: Example 2C.4

p.181: Hatcher writes (p.182)

> Since g is homotopic to the identity, it induces the identity across the top row of the diagram, and since g equals f near x, it induces the same map as f in the bottom row of the diagram, by excision.

Actually Hatcher is using not just the excision theorem, but the naturality of excision, which he hasn't explicitly stated. Here's a formulation of it:

> Let $f : (X, A) \to (Y, B)$ be a map of pairs. Let Z be excisive for (X, A), i.e., $\operatorname{cl} Z \subset \operatorname{int} A$, and let W be excisive for (Y, B). Assume $f(X - Z) \subset Y - W$. (Note that this, plus $f(A) \subset B$, imply $f(A - Z) \subset B - W$.) Then we have a commutative diagram
>
> $$\begin{tikzcd} H_n(X - Z, A - Z) \arrow[r, "\approx"] \arrow[d, "f_*"] & H_n(X, A) \arrow[d, "f_*"] \\ H_n(Y - W, B - W) \arrow[r, "\approx"] & H_n(Y, B) \end{tikzcd} \tag{35}$$

This follows readily from the proof of the excision theorem (p.124). In brief, barycentric subdivision allows us to replace a chain in $C_n(X)$ with one made up of "small simplicies", i.e., simplicies lying either in A or in $X - Z$. Passing to relative chains $C_n(X, A)$, only the simplicies in $X - Z$ matter, giving rise to the isomorphism $H_n(X, A) \approx H_n(X - Z, A - Z)$. Now applying f to a chain of "small simplicies" for (X, A, Z) results in a chain of "small simplicies" for (Y, B, W) by the assumptions about f. Chasing through all the algebra yields the diagram (35).

In Example 2C.4, we let $X = Y$ be the genus 3 surface, $A = X - \{x\}$ and $B = X - \{y\}$ (so $f(A) = B$), and let Z be the complement of the neighborhood of x referred to: i.e., $X - Z$ is a neighborhood of x and $f|(X - Z) \equiv g|(X - Z)$. Also let $W = f(Z)$. We have diagrams (35) for both f and g. Since $f \equiv g$ on $X - Z$, we have

$$f_* \equiv g_* : H_n(X - Z, A - Z) \to H_n(X - W, B - W)$$

But then the two diagrams give us

$$f_* \equiv g_* : H_n(X, A) \to H_n(X, B)$$

which is what Hatcher claimed.

14 Theorem 3.5; Cohomology of the Klein Bottle

p.203: The second part of Theorem 3.5 may at first seem a bit surprising: the cellular cochain complex is isomorphic to the dual of the cellular chain complex. What happened to Ext and all that?

However, the cellular cochain complex should be regarded as a nicer version of the singular cochain complex. The "shifting of torsion" occurs during the computation of cohomology. The Klein bottle furnishes a nice illustration. Recall (25) the cellular chain complex:

$$0 \longrightarrow \overset{W_2}{\mathbb{Z}} \xrightarrow{K \mapsto 2b} \overset{W_1}{\mathbb{Z} \oplus \mathbb{Z}} \xrightarrow{0} \overset{W_1}{\mathbb{Z}} \longrightarrow 0$$

Dualizing, we get:

$$0 \longleftarrow \overset{W_2^*}{\mathbb{Z}} \xleftarrow{b^* \mapsto 2K^*} \overset{W_1^*}{\mathbb{Z} \oplus \mathbb{Z}} \xleftarrow{0} \overset{W_0^*}{\mathbb{Z}} \longleftarrow 0 \qquad (36)$$

Here K^*, a^*, and b^* are the duals of the generators K, a, b. (For example, $b^*(ma + nb) = n$.) Cranking through the definitions shows that $a^* \mapsto 0$ and $b^* \mapsto 2K^*$ under the map $W_1^* \to W_2^*$. We conclude immediately that $H^2 = \mathbb{Z}_2$ and $H^1 = H^0 = \mathbb{Z}$, just what we get from Corollary 3.3 (p.196).

Note the every 2-cochain is a cocyle, because we have $W_2^* \to 0$. Obviously this is true more generally: the top-dimensional cochains in a finite-dimensional cell complex are all cocycles. This helps explain how we can have $H^n \neq 0$ even though $H_n = 0$.

15 §3.2: Example 3.9

p.208–209: Hatcher computes the cup product for the 2-dim CW complex X obtained by attaching a 2-cell to S^1 via the map $z \mapsto z^m$. He starts with $H^n(X;\mathbb{Z})$ and $H^n(X, \mathbb{Z}_m)$, which can be computed in two ways. Look at the cellular chain complexes, $C_n(X, \mathbb{Z})$ and $C_n(X, \mathbb{Z}_m)$:

$$\begin{array}{ccccccccc} & & C_2 & & C_1 & & C_0 & & \\ \\ 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{m} & \mathbb{Z} & \xrightarrow{0} & \mathbb{Z} & \longrightarrow & 0 \\ \\ 0 & \longrightarrow & \mathbb{Z}_m & \xrightarrow{0} & \mathbb{Z}_m & \xrightarrow{0} & \mathbb{Z}_m & \longrightarrow & 0 \end{array} \tag{37}$$

This gives us the homology groups with $\mathbb{Z}$ and $\mathbb{Z}_m$ coefficients:

	H_2	H_1	H_0
$\mathbb{Z}$:	0	$\mathbb{Z}_m$	$\mathbb{Z}$
$\mathbb{Z}_m$:	$\mathbb{Z}_m$	$\mathbb{Z}_m$	$\mathbb{Z}_m$

Note how $H_2(X; \mathbb{Z}_m)$ escapes being 0 because the “multiply by m” map in $\mathbb{Z}$ coefficients has become a 0 map.

One way to find the cohomology groups is to dualize (37), using Theorem 3.5 (p.203). Here is $\mathrm{Hom}(C_n, \mathbb{Z})$, and below it $\mathrm{Hom}(C_n, \mathbb{Z}_m)$:

$$\begin{array}{ccccccccc} & & C^2 & & C^1 & & C^0 & & \\ & & & & & & & & \\ 0 & \longleftarrow & \mathbb{Z} & \xleftarrow{m} & \mathbb{Z} & \xleftarrow{0} & \mathbb{Z} & \longleftarrow & 0 \\ & & & & & & & & \\ 0 & \longleftarrow & \mathbb{Z}_m & \xleftarrow{0} & \mathbb{Z}_m & \xleftarrow{0} & \mathbb{Z}_m & \longleftarrow & 0 \end{array} \tag{38}$$

For the middle line, these are all $\mathrm{Hom}(C_n(X;\mathbb{Z}),\mathbb{Z})$, since $\mathrm{Hom}(\mathbb{Z}_m,\mathbb{Z}) = 0$. For the bottom line, it could be $\mathrm{Hom}(C_n(X;\mathbb{Z}),\mathbb{Z}_m)$ or $\mathrm{Hom}(C_n(X;\mathbb{Z}_m),\mathbb{Z}_m)$ — it doesn't matter, we get the same results! However, Theorem 3.5, as stated, applies to $\mathrm{Hom}(C_n(X;\mathbb{Z}),\mathbb{Z}_m)$.

So the cohomology groups are:

	H^2	H^1	H^0
$\mathbb{Z}$:	$\mathbb{Z}_m$	0	$\mathbb{Z}$
$\mathbb{Z}_m$:	$\mathbb{Z}_m$	$\mathbb{Z}_m$	$\mathbb{Z}_m$

We can also use Corollary 3.3 (p.196) for $H^n(X;\mathbb{Z})$: $H^n(X;\mathbb{Z}) = (H_n/T_n)\oplus T_{n-1}$, where T_n, T_{n-1} are the torsion subgroups. For $H^n(X;\mathbb{Z}_m)$, we use Theorem 3.2 (p.195): $H^n(X;\mathbb{Z}_m) = \mathrm{Hom}(H_n(X),\mathbb{Z}_m)\oplus\mathrm{Ext}(H_{n-1}(X),\mathbb{Z}_m)$. Note that $\mathrm{Ext}(\mathbb{Z}_m,\mathbb{Z}_m) = \mathbb{Z}_m$ and $\mathrm{Ext}(\mathbb{Z},\mathbb{Z}_m) = 0$ by the bottom of p.195.

In the middle of p.209, Hatcher mentions that the map $h : H^2(X;\mathbb{Z}_m) \to \mathrm{Hom}(H_2(X;\mathbb{Z}_m),\mathbb{Z}_m)$ is an isomorphism. This h is *not* the h from the universal coefficient theorem, defined on p.191. The "universal coefficient" h looks like this:

$$h : H^2(X;\mathbb{Z}_m) \to \mathrm{Hom}(H_2(X;\mathbb{Z}),\mathbb{Z}_m)$$

i.e.,

$$h : \mathbb{Z}_m \to \mathrm{Hom}(0, \mathbb{Z}_m) = 0$$

not an isomorphism. Rather than trying to figure out a connection to the "universal coefficient" h, let's see what Hatcher needs to finish the example. We know that $H_2(X; \mathbb{Z}_m) = H^2(X; \mathbb{Z}_m) = \mathbb{Z}_m$, and since $\mathrm{Hom}(\mathbb{Z}_m, \mathbb{Z}_m) = \mathbb{Z}_m$, the two groups in question are both $\mathbb{Z}_m$. Hatcher needs a generator for $H^2(X; \mathbb{Z}_m)$ to compare to $\varphi \smile \varphi$.

A $\mathbb{Z}_m$ 2-cocycle is an element of $\mathrm{Hom}(C_2(X), \mathbb{Z}_m)$. At the cellular level, $C_2(X)$ is generated by the unique 2-cell of X; let's call the generator τ. With $\mathbb{Z}$ coefficients, τ is not a cycle . But that doesn't matter, since we want an element of $\mathrm{Hom}(C_2(X), \mathbb{Z}_m)$.

Let $\tau^* \in \mathrm{Hom}(C_2(X), \mathbb{Z}_m)$ be the dual cochain: i.e., $\tau^*(\tau) = 1$. Because $\dim X = 2$, 2-cochains are automatically 2-cocyles, and it's not hard to see that τ^* (or rather its cohomology class) generates $H^2(X; \mathbb{Z}_m)$.

We now transfer to the Δ-complex description. Here τ is represented by $\sum_i T_i$. Since the T_i form a basis for 2-chains, we specify elements of $\mathrm{Hom}(C_2(X), \mathbb{Z}_m)$ by giving their values on all the T_i. We want $\tau^*(\sum_i T_i) = 1$. If we achieve that, we're golden. The obvious choice is to let τ^* take the value 1 on one of the T_i, and 0 on all the others.

It is worth noting that the cellular structure provides the easiest computation of the cohomology groups, but you have to go to the Δ-complex to compute the ring operation (cup product).

16 §3.3: Poincaré Duality

p.232–233: Hatcher outlines the following geometric argument for Poincaré duality. Suppose X and $\overline{X}$ are dual cell structures on a compact orientable n-dimensional manifold. Since the manifold is compact, there are only finitely many cells in X and $\overline{X}$. We consider three chain complexes: $C(X)$ = chains on X; $C(\overline{X})$ = chains on $\overline{X}$, but "written backwards" (see below); and $C^*(X)$ = cochains on X. Thus:

$$
\begin{gathered}
0 \longrightarrow C_n(X) \xrightarrow{\partial} C_{n-1}(X) \xrightarrow{\partial} \cdots \xrightarrow{\partial} C_1(X) \xrightarrow{\partial} C_0(X) \longrightarrow 0 \\[2em]
0 \longleftarrow C_0(\overline{X}) \xleftarrow{\partial} C_1(\overline{X}) \xleftarrow{\partial} \cdots \xleftarrow{\partial} C_{n-1}(\overline{X}) \xleftarrow{\partial} C_n(\overline{X}) \longleftarrow 0 \\[2em]
0 \longleftarrow C_0^*(X) \xleftarrow{\delta} C_1^*(X) \xleftarrow{\delta} \cdots \xleftarrow{\delta} C_{n-1}^*(X) \xleftarrow{\delta} C_n^*(X) \longleftarrow 0
\end{gathered}
\tag{39}
$$

For any k, we have distinguished bases for $C_k(X)$, $C_k^*(X)$, and $C_{n-k}(\overline{X})$, enabling us to define abelian group isomorphisms $C_k(X) \cong C_k^*(X) \cong C_{n-k}(\overline{X})$. Namely, let $\{e_1, \ldots, e_M\}$ be the k-cells of X; let $\overline{e}_i$ be the $(n-k)$-cell in $\overline{X}$ dual to e_i; let e_i^* be the linear functional in $C_k^*(X)$ that is 1 on e_i and 0 on all the other e_j: $e_i^*(e_j) = \delta_{ij}$. The isomorphisms are

$$
\sum_i c_i e_i \leftrightarrow \sum_i c_i e_i^* \leftrightarrow \sum_i c_i \overline{e}_i \tag{40}
$$

so we have a diagram

$$
\begin{tikzcd}
0 & C_n^*(X) \arrow[l] \arrow[d, leftrightsquigarrow] & C_{n-1}^*(X) \arrow[l, "\delta"'] \arrow[d, leftrightsquigarrow] & \cdots \arrow[l, "\delta"'] & C_1^*(X) \arrow[l, "\delta"'] \arrow[d, leftrightsquigarrow] & C_0^*(X) \arrow[l, "\delta"'] \arrow[d, leftrightsquigarrow] & 0 \arrow[l] \\
0 \arrow[r] & C_n(X) \arrow[r, "\partial"] \arrow[d, leftrightsquigarrow] & C_{n-1}(X) \arrow[r, "\partial"] \arrow[d, leftrightsquigarrow] & \cdots \arrow[r, "\partial"] & C_1(X) \arrow[r, "\partial"] \arrow[d, leftrightsquigarrow] & C_0(X) \arrow[r] \arrow[d, leftrightsquigarrow] & 0 \\
0 & C_0(\overline{X}) \arrow[l] & C_1(\overline{X}) \arrow[l, "\partial"'] & \cdots \arrow[l, "\partial"'] & C_{n-1}(\overline{X}) \arrow[l, "\partial"'] & C_n(\overline{X}) \arrow[l, "\partial"'] & 0 \arrow[l]
\end{tikzcd}
\tag{41}
$$

Unfortunately, this diagram does *not* commute. (I use squiggly vertical arrows to indicate the lack of commutativity.) But if we omit the middle row, the resulting diagram *does* commute:

$$\begin{tikzcd}
0 & C_n^*(X) \arrow[l] \arrow[d, leftrightarrow] & C_{n-1}^*(X) \arrow[l, "\delta"'] \arrow[d, leftrightarrow] & \cdots \arrow[l, "\delta"'] & C_1^*(X) \arrow[l, "\delta"'] \arrow[d, leftrightarrow] & C_0^*(X) \arrow[l, "\delta"'] \arrow[d, leftrightarrow] & 0 \arrow[l] \\
0 & C_0(\overline{X}) \arrow[l] & C_1(\overline{X}) \arrow[l, "\partial"'] & \cdots \arrow[l, "\partial"'] & C_{n-1}(\overline{X}) \arrow[l, "\partial"'] & C_n(\overline{X}) \arrow[l, "\partial"'] & 0 \arrow[l]
\end{tikzcd} \tag{42}$$

Once we have that, we immediately get $H^k \cong H_{n-k}$, since the homology and cohomology groups are independent of the choice of cell structure. This is Hatcher's version of Poincaré Duality.

The commutativity argument is straightforward, once we set up a little notation. Let $\{f_1, \ldots, f_N\}$ be the $(k-1)$-cells of X. Let ι_{ij} be the "incidence coefficients" describing the boundaries of k-cells:

$$\partial e_i = \sum_{j=1}^{N} \iota_{ij} f_j \tag{43}$$

So $\iota_{ij} = 0$ if f_j is not a face of e_i, otherwise $\iota_{ij} = \pm 1$, depending on orientations.

The duality of X with $\overline{X}$ gives (indeed, *means*) that

$$\partial \overline{f}_j = \sum_{i=1}^{M} \iota_{ij} \overline{e}_i \tag{44}$$

that is, $\overline{f}_j$ has $\overline{e}_i$ as a face if and only if e_i has f_j as a face, moreover with the same orientation sign. In other words, (44) is part of what we mean by a "dual cell complex". Regarding this, Hatcher says:

> ...the problem of signs must be addressed. After analyzing the situation more closely, one sees that if M [my X] is orientable, it is possible to make consistent choices of orientations of all the cells of C and C^* [my $C(\overline{X})$] so that the boundary maps in C agree with the coboundary maps in C^* ...

I will defer this matter till later.

To establish commutativity, then, we need this analog for (44):

$$\delta f_j^* = \sum_{i=1}^{M} \iota_{ij} e_i^* \tag{45}$$

For (44) and (45) then tell us that $\partial \overline{f}_j$ and δf_j^* correspond. By linearity we then conclude that (42) commutes.

By linearity, it's enough to check (45) on basis elements, i.e., $\delta f_j^*(e_l) = \left(\sum_{i=1}^{M} \iota_{ij} e_i^*\right) e_l$. Now, δ is defined by the formula

$$\delta\varphi(\alpha) = \varphi(\partial\alpha), \quad \varphi \in C_{k-1}^*(X), \alpha \in C_k(X) \tag{46}$$

So

$$\begin{aligned} \delta f_j^*(e_l) &= f_j^*(\partial e_l) \\ &= f_j^*\left(\sum_{i=1}^{N} \iota_{li} f_i\right) \\ &= \sum_{i=1}^{N} \iota_{li} f_j^* f_i \\ &= \iota_{lj} \end{aligned} \tag{47}$$

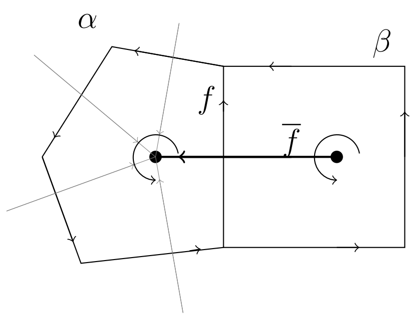
However, if we apply $\sum_{i=1}^{M} \iota_{ij} e_i^*$ to e_l we get the same result:

$$\left(\sum_{i=1}^{M} \iota_{ij} e_i^*\right) e_l = \sum_{i=1}^{M} (\iota_{ij} e_i^* e_l) = \iota_{lj} \tag{48}$$

QED.

Underlying the algebra is a simple idea, shown in figure 4. Consider the pentagonal 2-cell α in $C(X)$ and its boundary, which is a (signed) sum of 1-cells. Select one of these 1-cells, say the one labeled f in the picture. Its

Figure 4: Dual Cells



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dual $\overline{f}$ in $\overline{X}$ is a 1-cell piercing f. The boundary of $\overline{f}$ is a (signed) sum of two points, one inside α (with $+$), the other in the neighboring cell β (with $-$). (We will look at the orientation choices shortly.)

Now consider a 1-cochain f^* "concentrated" on f, taking the value 1 on f and 0 everywhere else. Form δf^*. This is a 2-cochain taking the value 1 on α and -1 on β. We can see this directly from the definition (46) of δ: $\delta f^*(\alpha) = f^*(\partial\alpha)$, so by linearity we apply f^* to each of the five 1-cells of $\partial\alpha$ and sum the results (with appropriate signs). Likewise, $\delta f^*(\beta) = -1$.

So δf^* and $\partial\overline{f}$ "look the same"; more precisely, the diagram (42) commutes. We can also see how diagram (41) fails to commute. Here is an excerpt from

(41) for the case $n = 2$:

$$\begin{tikzcd} 0 \arrow[r] & \alpha \in C_2(X) \arrow[r, "\partial"] \arrow[d, leftrightsquigarrow] & \partial\alpha \in C_1(X) \arrow[r, "\partial"] \arrow[d, leftrightsquigarrow] & \cdots \\ 0 & \overline{\alpha}, \partial\overline{\partial\alpha} \in C_0(\overline{X}) \arrow[l] & \overline{\partial\alpha} \in C_1(\overline{X}) \arrow[l, "\partial"'] & \cdots \arrow[l, "\partial"'] \end{tikzcd} \tag{49}$$

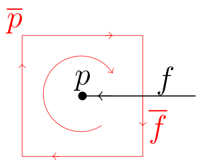
The vertical lines are the dualizing operator $\overline{}$. On the one hand, $\overline{\alpha}$ is just the point (0-cell) inside α; on the other hand, $\overline{\partial\alpha}$ is the "star" of edges piercing all the sides of α, and $\partial\overline{\partial\alpha}$ is a sum of six points, namely one point in each of the 2-cells bordering the pentagon (with coefficient -1) and the point in the pentagon (with coefficient 5).

Since (42) commutes, and the bottom two lines of (41) fail to commute, it follows that the top two line of (41) must also fail to commute.

Finally we return to the matter of orientations. Let's assume we have an *unoriented* dual cell structure. Our original cell structure is oriented: this means that each cell e_i and f_j has an orientation, determining the incidence coefficient ι_{ij}. We need to discuss:

1. What we mean by an orientation of a cell.
2. How the orientations determine the incidence coefficients, and thus the boundary operator via (43).
3. How to choose the orientations of the dual cells so as to insure (44).

Figure 5: Duality for 2-Cells



16.1 Low-Dimensional Cases

To develop some intuition, we start off with the 2d case. We know how orientation and incidence works here: arrows, clockwise/counterclockwise, and incidence determined by agreement. In figure 4, α and β have the usual (positive) counterclockwise orientation, so $\iota(\alpha, f) = 1$ and $\iota(\beta, f) = -1$ (adapting the ι notation). What about orienting the dual cells?

In the same figure, $\overline{f}$ points from $\overline{\beta}$ to $\overline{\alpha}$, so that $\iota(\overline{f}, \overline{\alpha}) = 1$ and $\iota(\overline{f}, \overline{\beta}) = -1$, as required by duality. It looks like we have to rotate an oriented 1-cell $90°$ counterclockwise to get the dual cell. (Note: this rule means dualizing is not symmetric: $\overline{\overline{f}} = -f$.)

Our rule for $f \to \overline{f}$ implicitly assumes a consistent notion of "counter-clockwise" across the entire surface, i.e., orientability. We lean on this to handle 0-cells and 2-cells. We adopt the convention that all 0-cells and 2-cells in $C(X)$ have positive orientation. Now look at figure 5. In black we have a 0-cell p and a 1-cell f, and in red we have the dual 2-cell $\overline{p}$ and 1-cell $\overline{f}$. We are forced to assign $\overline{p}$ a negative (clockwise) orientation, to get $\iota(f, p) = \iota(\overline{p}, \overline{f}) = 1$. Upshot: all dual 0-cells have positive orientation, all dual 2-cells have negative orientation.

The 3d discussion is similar but more complicated. The usual right-hand rule determines incidence for 3-cells and 2-cells: if Φ is a 3-cell and α is a 2-cell, then we decorate Φ with outward vectors, one for each face. If the "circulation" in α determined by the right-hand rule agrees with the orientation of α, then $\iota(\Phi, \alpha) = 1$, otherwise $\iota(\Phi, \alpha) = -1$.

Here are dualizing rules for 3d:

- All dual 0-cells and 3-cells are given positive orientation.
- If α is a 2-cell, then $\overline{\alpha}$ is a 1-cell. The orientation of $\overline{\alpha}$ is given by the *left-hand rule.*
- If f is a 1-cell, then $\overline{f}$ is a 2-cell. The orientation of $\overline{f}$ is given by the *left-hand rule.*

The last two bullets may surprise you; I leave it as an exercise to verify them.

16.2 Orientation

We now turn to the general case. Earlier we let X be a cell structure on an n-dimensional manifold; we'll abuse notation slightly and also let X denote the underlying manifold. We have an atlas of coordinate charts covering X; if two charts φ and η overlap, then we can form the composition $\eta^{-1}\varphi$, a map from one open ball in $\mathbb{R}^n$ to another. For a differential manifold, $\eta^{-1}\varphi$ must be a diffeomorphism; for a topological manifold, merely a homeomorphism. This compatibility condition is a cornerstone of manifold theory. Starting with some atlas, we can enlarge to the maximal atlas of all compatible coordinate charts: call that $\mathcal{A}$.

Diffeomorphisms from one open n-ball to another (or equivalently, from the unit open n-ball to itself) are classified as orientation preserving or reversing. Thus: look at the derivative at a point, a linear map that will have a positive or negative determinant, and then note that this positive/negative character must stay the same for the entire ball by continuity. For general homeomorphisms instead of diffeomorphisms, you can do this classification by appealing to the local homology groups; Hatcher outlines this on p.233–234.

Let's return to our manifold X with maximal atlas $\mathcal{A}$. Can $\mathcal{A}$ be divided into two subatlases, say $\mathcal{A}^+$ and $\mathcal{A}^-$, such that any two overlapping charts in the same subatlas will have an orientation preserving composition? (I.e.: if $\varphi, \eta \in \mathcal{A}^+$, and if φ, η overlap, then $\eta^{-1}\varphi$ is orientation preserving; likewise for $\mathcal{A}^-$.) If so, we say X is **orientable**. If X is connected (which we assume from now on), then the decomposition $\mathcal{A} = \mathcal{A}^+ \cup \mathcal{A}^-$ is unique. Each of $\mathcal{A}^+$ and $\mathcal{A}^-$ is called an **orientation**.

Orientation of cells: For any k-cell, we have a homeomorphism between it and the open k-ball. Mimicking the above discussion for coordinate charts, we look pairs of such homeomorphisms φ, η : open k-ball $\to e$, then at whether $\eta^{-1}\varphi$ is orientation preserving or reversing, and finally end up with two orientations for the cell, each given by a class of homeomorphisms from the k-ball to the cell.

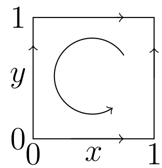
Let's see how this plays out in 2d and 3d. An open 1-cell has two classes of homeomorphisms from the open interval $(-1, 1)$ (the open 1-ball), clearly the two directions one can use when parametrizing the cell. Thus the arrow on the arc. An open 2-cell is the homeomorphic image of the open disk; we pass from one class of homeomorphisms to the other by composing with a reflection of the unit disk, such as $(x, y) \mapsto (y, x)$ or $(x, y) \mapsto (-x, y)$. The "sense of rotation", indicated by the arrow-tipped circular arc, clearly tracks this. Finally, an open 3-cell is the homeomorphic image of the open ball; an oriented basis, or a glove, or the right-hand/left-hand rule all serve to track orientation.

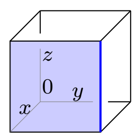
If X is an oriented n-manifold, with orientation $\mathcal{A}^+$, we have a well-defined way to assign positive orientations to all the n-cells: just make things agree with the charts. But for $k < n$, that doesn't work. For example, given a cell structure for an oriented surface, we could assign the counterclockwise orientation to all 2-cells, but the arrows have to be assigned to each arc arbitrarily. Indeed, as figure 4 shows, the formula $\partial\partial = 0$ depends on this "mismatch" of incidences between neighboring 2-cells.

16.3 Incidence coefficients

First, a technical point. As figure 3 shows, the definition of a cell complex allows a lot of leeway in attaching cells. We want incidence coefficients to follow naive intuition. Roughly, we want a simplicial complex, but modified to permit polygons, polyhedrons, etc., in place of simplices. I won't spell out the details, since the purpose of these notes is intuition and not formal proof. Let us also assume that each closed cell is contained in a coordinate chart.

Our strategy: first define when a k-cell and $(k-1)$-cell have $\iota = 1$, then bootstrap to define ι in general. We use as a model the picture at right. The cells are given orientations inherited from the standard coordinates. For example, in the square, $\{(1, y)|0 < y < 1\}$ parametrizes the right vertical edge. In the cube, $\{(1, y, z)|0 < y < 1, 0 < z < 1\}$ parametrizes the front face. Both of these have $\iota = 1$ with the top-dimensional cell. Furthermore, in the cube, $\{(1, 1, z)|0 < z < 1\}$ parametrizes the right vertical edge of the front face, and that has $\iota = 1$ with the front face.





This suggests the following definition:

> Let e be a k-cell, f one of its faces, and let p be a point in the interior of f. Suppose there is a coordinate chart in $\mathcal{A}^+$ such that, in a neighborhood of p, e is parametrized by $\{(\overbrace{1, \ldots, 1}^{n-k}, y, z_1, \ldots, z_{k-1})|0 <$

$y < 1, (\forall j) 0 < z_j < 1\}$, and f is parametrized by $\{(\overbrace{1, \ldots, 1}^{n-k+1}, z_1, \ldots, z_{k-1}) | (\forall j$ $z_j < 1\}$. If these parametrizations both agree with the chosen orientations of e and f, then $\iota(e, f) = 1$.

It's worth checking that this definition accords with the model examples. A full justification of this definition would take some work. For example, we'd need to establish independence of the choice of point and chart.

We sum this up as follows:

$$e : \begin{array}{|c|c|c|} \hline 1, \ldots, 1 & 0 < y < 1 & 0 < \vec{z} < 1 \\ \hline \end{array}$$

$$f : \begin{array}{|c|c|c|} \hline 1, \ldots, 1 & 1 & 0 < \vec{z} < 1 \\ \hline \end{array}$$

If we flip either of the chosen orientations of e and f, then ι should flip sign. We could indicate this with the following notation:

$$\iota(e, f) = o(e)o(f)$$

$$e : \begin{array}{|c|c|c|} \hline 1, \ldots, 1 & 0 < y < 1 & 0 < \vec{z} < 1 \\ \hline \end{array}$$

$$f : \begin{array}{|c|c|c|} \hline 1, \ldots, 1 & 1 & 0 < \vec{z} < 1 \\ \hline \end{array}$$

where $o(e)$ and $o(f)$ are $+/-$ if the inherited parametrizations agree/disagree with the orientations of e and f. If one is willing to take on faith the existence of a suitable chart, then we have a definition for the incidence coefficients in all cases.

We need to say a few words about "inherited parametrizations". There is an obvious bijection $(y, z_1, \ldots, z_{k-1}) \mapsto (1, \ldots, 1, y, z_1, \ldots, z_{k-1})$, thus composing with the chart $(1, \ldots, 1, y, z_1, \ldots, z_{k-1}) \mapsto$ point $\in M$, we obtain a homeomorphism between the open k-ball and the cell e. The agreement/disagreement of this homeomorphism with the chosen orientation of e determines the incidence coefficient.

Immediate consequence: it doesn't matter where we put the first block of 1's in our discussion, or indeed what constants we use. So we also have the rule:

$$\iota(e,f) = o(e)o(f)$$

$$e: \boxed{0<y<1}\boxed{0<\vec{z}<1}\boxed{1,\ldots,1}$$

$$f: \boxed{\quad 1 \quad}\boxed{0<\vec{z}<1}\boxed{1,\ldots,1}$$

For the next step, we will need one more rule:

$$\iota(e,f) = (-1)^{\ell+1}o(e)o(f)$$

$$e: \boxed{0<x_1,\ldots,x_\ell<1}\boxed{0<y<1}\boxed{1,\ldots,1}$$

$$f: \boxed{0<x_1,\ldots,x_\ell<1}\boxed{\quad 0 \quad}\boxed{1,\ldots,1}$$

We've switched notation from z to x, and from k to ℓ. Also, $0<x_1,\ldots,x_\ell<1$ is shorthand for $(x_1,\ldots,x_\ell)$ with $0<x_i<1$ for all i. In the next section we will assume that $k+\ell=n$; at this point, ℓ is any number from 0 to n.

To derive this from the preceding rule, consider the bijection from $\mathbb{R}^{\ell+1}$ to itself $(y,x_1,\ldots,x_\ell)\mapsto(x_1,\ldots,x_\ell,-y)$. This is a linear map with determinant $(-1)^{\ell+1}$. Composing with the parametrization taking $(x_1,\ldots,x_\ell,y)$ into X, we have a parametrization of e that looks like this: $e: \boxed{0<x_1,\ldots,x_\ell<1}\boxed{-1<y<0}\boxed{1,\ldots,1}$. The face f is given by setting $y=-1$. A simple translation changes this to $e: \boxed{0<x_1,\ldots,x_\ell<1}\boxed{0<y<1}\boxed{1,\ldots,1}$, with f obtained by setting $y=0$. So we've introduced a factor of $(-1)^{\ell+1}$ into $o(e)$. On the other hand, for f, we've just chosen a different spot and value for one of the constants, and as we've seen, this makes no difference.

Finally, throwing in a translation and a stretching, and using different constants, we end up with the two rules needed for the next section.

$$\iota(e,f) = o(e)o(f)$$

$$e: \begin{array}{|c|c|c|}\hline 2,\ldots,2 & 0<y<2 & 0<z_1,\ldots,z_{k-1}<2 \\ \hline\end{array}$$

$$f: \begin{array}{|c|c|c|}\hline 2,\ldots,2 & 2 & 0<z_1,\ldots,z_{k-1}<2 \\ \hline\end{array}$$

and

$$\iota(e,f) = (-1)^{\ell+1}o(e)o(f)$$

$$e: \begin{array}{|c|c|c|}\hline 1<x_1,\ldots,x_\ell<3 & 1<y<3 & 1,\ldots,1 \\ \hline\end{array}$$

$$f: \begin{array}{|c|c|c|}\hline 1<x_1,\ldots,x_\ell<3 & 1 & 1,\ldots,1 \\ \hline\end{array}$$

Concluding remark: we have not yet made use of the orientability assumption. We did require our coordinate chart to belong to $\mathcal{A}^+$, but this was inessential. Intuitively, the incidence coefficient between an edge and a polygon shouldn't depend on their embedding in a higher dimensional space. I leave it as an exercise to puzzle out how this works, in terms of the parametrizations.

16.4 Dualization

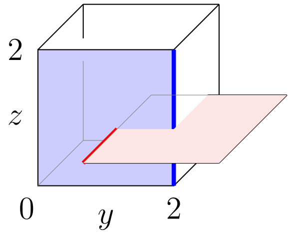
Our starting point is the setup described above:

> Let e be a k-cell, f one of its faces, and let p be a point in the interior of f. Suppose there is a coordinate chart in $\mathcal{A}^+$ such that, in a neighborhood of p, e and f are parametrized by

$$e: \begin{array}{|c|c|c|}\hline 2,\ldots,2 & 0<y<2 & 0<z_1,\ldots,z_{k-1}<2 \\ \hline\end{array}$$

$$f: \begin{array}{|c|c|c|}\hline 2,\ldots,2 & 2 & 0<z_1,\ldots,z_{k-1}<2 \\ \hline\end{array}$$

Figure 6: Dualization



We will begin by defining provisional dual cells $\ddot{e}$ and $\ddot{f}$, or actually just near p. Our initial choice of orientation will not preserve incidence coefficients, so we will then modify it to give the real $\overline{e}$ and $\overline{f}$.

Remember we assume that an *unoriented* dual cell structure has already been constructed. Our only goal is to assign orientations, and this we can while sticking close to p. Our discussion still falls well short of a rigorous proof, but the purpose of these note is intuition.

To make life easier, let's assume $o(e) = o(f) = 1$, i.e., the orientations of e and f agree with the parametrizations given above. We now give the parametrizations for $\ddot{f}$ and $\ddot{e}$, and repeat the ones for e and f, for convenience:

$\ddot{f}$:	$1 < x_1, \ldots, x_\ell < 3$	$1 < y < 3$	$1, \ldots, 1$
$\ddot{e}$:	$1 < x_1, \ldots, x_\ell < 3$	1	$1, \ldots, 1$
e :	$2, \ldots, 2$	$0 < y < 2$	$0 < z_1, \ldots, z_{k-1} < 2$
f :	$2, \ldots, 2$	2	$0 < z_1, \ldots, z_{k-1} < 2$

Here, $k+\ell=n$, something we assume throughout this section. To convince yourself of the correctness of these parametrizations, examine figure 6. We have two shaded squares and two edges (marked by thickness and color); the blue square is dual to the red edge, and the blue edge is dual to the red square.

Note the essential restriction to a chart in $\mathcal{A}^+$. For example, if we switch the direction of an x_i-axis, but make no other change, this preserves the orientation of e but switches the orientation of $\ddot{e}$.

Now consider a sequence of cells in descending dimensions, $e^n, \ldots, e^0$. Suppose all the incidence coefficients are 1: $\iota(e^k, e^{k-1}) = 1$. From the rules of the last section, we know that $\iota(\ddot{e}^{k-1}, \ddot{e}^k) = (-1)^{\ell+1}$, where $k+\ell=n$. Note that ℓ is the dimension of $\ddot{e}^k$. So we have a sequence of dual cells in ascending dimensions with alternating incidence coefficients:

$$\begin{array}{rcccccccccccccc} \text{dim}: & 0 & & 1 & & 2 & & 3 & & 4 & & 5 & & \ldots & n \\ \iota: & & - & & + & & - & & + & & - & & + & \ldots & \end{array}$$

We can fix this by changing the orientations of the $\ddot{e}^k$ with the pattern $+--++--+\cdots$:

$$\begin{array}{ccccccccccccc} +0 & & -1 & & -2 & & +3 & & +4 & & -5 & & \ldots \\ & + & & + & & + & & + & & + & & + & \ldots \end{array}$$

(Where $\pm\ell$ is short for "the dual cell $\ddot{e}^k$ of dimension ℓ with the orientation multiplied by ± 1".)

We've imagined a whole sequence of cells, but that was just to make the pattern clear. A little reflection should make it clear that if we set $\overline{e} = \pm\ddot{e}$, with the $\pm$ given by the $+--++--+\cdots$ pattern, then incidence coefficients are preserved: $\iota(\overline{f}, \overline{e}) = \iota(e, f)$.

So we are done. We had to work pretty hard just to get a plausibility argument for Poincaré duality. No wonder Hatcher took a different approach!